DEPARTMENT OF APPLIED STATISTICS
(COMPUTING SECTION)
UNIVERSITY OF LONDON, UNIVERSITY COLLEGE

TRACTS FOR COMPUTERS

EDITED BY KARL PEARSON, F.R.S.

No. VIII

Table of the Logarithms of the Complete Γ-Function (for Arguments 2 to 1200, i.e. beyond Legendre's Range)

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TRACTS FOR COMPUTERS

No. VIII

CONTENTS

THE COMPLETE Γ-FUNCTION

By Egon S. Pearson, B.A.

I. *Methods of Calculation.*

The publication of the Tables of the Incomplete Gamma Function*, or rather of the ratio

$$I_x(p) = \int_0^x e^{-x} x^{p-1} dx \Big/ \int_0^\infty e^{-x} x^{p-1} dx = \int_0^x e^{-x} x^{p-1} dx \Big/ \Gamma(p),$$

renders desirable the issue of a Table of the Complete Gamma Function in readily accessible form. In many statistical problems the value of $\Gamma(p)$ is required for high values of p, amounting perhaps to several hundreds, and in such cases it is practically essential to deal with Log $\Gamma(p)$; it is this function which is tabled below. The most important Tables of Log $\Gamma(p)$ that have been published are:

(i) That of Gauss, first issued in 1813† and to be found in his *Werke*, Band III. s. 161–162, Göttingen 1866; this Table gives to twenty places of decimals the value of Log $\Gamma(p)$ in the range 1·00 to 2·00 for an argument interval of 0·01;

(ii) That of Legendre originally published in Tom. II. (pp. 490–499) of his *Traité des Fonctions Elliptiques, avec des Tables pour en faciliter le calcul numérique,* Paris 1825; this Table is calculated to twelve places of decimals for the range 1·000 to 2·000 with an argument interval of 0·001. A facsimile reproduction has been issued as No. IV of the present series of Tracts for Computers.

The value of Log $\Gamma(p)$ for values of p lying outside the range of these two tables may be calculated by various methods:

(a) By successive use of the reduction relation

$$\Gamma(p) = (p-1)\,\Gamma(p-1),$$

a laborious process if p is much above 2.

* Tables of the Incomplete Gamma Function; issued by His Majesty's Stationery Office, 1922.

† Disquisitiones Generales Circa Seriem Infinitam

$$1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1)\,\beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} x^2 + \ldots.$$

Commentationes Societatis regiae Scientiarum Gottingensis recentiores, Vol. II. Gottingae MD.CCCXIII.

(*b*) By Stirling's asymptotic expansion

$$\operatorname{Log}_e \Gamma(p) = (p - \tfrac{1}{2}) \log_e p - p + \tfrac{1}{2} \log_e 2\pi + \frac{1}{12p} - \frac{1}{360p^3} + \frac{1}{1260p^5} - \frac{1}{1680p^7} + \frac{1}{1188p^9} - \dots$$

(*c*) By Professor Forsyth's formula

$$\Gamma(p+1) = \sqrt{2\pi} \left\{ \frac{\sqrt{p^2 + p + \frac{1}{6}}}{e} \right\}^{p+\frac{1}{2}},$$

which may be used for calculation of Log $\Gamma(p+1)$ to 5 or 6 places of decimals.

(*d*) By Professor Pearson's formula*

$$\operatorname{Log} \frac{\Gamma(p+1)}{p^p e^{-p}} = 0{\cdot}399{,}0899 + \tfrac{1}{2} \log p + {\cdot}080{,}929 \sin \frac{25^\circ{\cdot}623}{p},$$

which is particularly convenient in many statistical problems where it is the ratio of $\Gamma(p+1)$ to $p^p e^{-p}$ that occurs.

(*e*) By interpolation in a Table of the Logarithms of factorials†.

These methods are none of them simple, and may be extremely laborious if a high degree of accuracy is required. It is hoped, therefore, that the computer's problem will be simplified by the issue of the present Table containing the values of Log $\Gamma(p)$ to ten places of decimals from $p = 2$ to $p = 1200$, together with central differences for purposes of interpolation‡.

* *Biometrika*, Vol. VI. pp. 118–119.

† As for example Degen's Table of the Logarithms of factorials up to 1200, calculated to eighteen decimal places. *Tabularum ad faciliorem et breviorem probabilitatis computationem utilium Enneas.* Havniae MDCCCXXIV. The Logarithms of factorials up to 1000 to seven decimal places are also given in *Tables for Statisticians and Biometricians.* Cambridge University Press.

‡ To make full use of the present Table, ten figure logarithms are of course required. These are unfortunately not readily accessible. Vega's *Thesaurus Logarithmorum completus*, Leipzig, 1794, is now a rare book; it also contains certain last figure errors. Andoyer, in the introduction to his *Nouvelles Tables Trigonométriques Fondamentales*, states that photozincographic reproductions of Vega's Tables were issued by the Instituto Geografico Militare at Florence in 1889, 1892 and 1910; the original errors are reproduced in these facsimile reprints. Duffield's ten figure Table calculated independently of Vega's, and issued for the U.S. Coast and Geodetic Survey, Washington, 1897, has been long out of print.

Briggs' Table (to fourteen figures) may be used, and there are several other tables going to more than ten figures, but their range is small and lengthy interpolation may be required in using them.

II. *Construction and Scope of the present Table.*

The Table has been constructed as follows:

(i) The values of Log $\Gamma(p)$ for the range $p = 2$ to $p = 70$ were obtained by method (a) above, by the successive addition of the logarithms of ordinary numbers to the logarithms of $\Gamma(p)$ (range $p = 1$ to $p = 2$). The work was first carried out starting from Legendre's twelve figure Table*, but as it was found that in certain cases this led to a unit error in the tenth place of a 4th or 6th difference, the results were checked by working to fifteen places of decimals from Gauss' Table, and finally cutting off to ten places. This portion of the Table is divided into two parts: from $p = 2{\cdot}0$ to $5{\cdot}0$, the argument interval is 0·1, while from $p = 5{\cdot}0$ to $70{\cdot}0$ it is 0·2.

(ii) For the range $p = 70$ to $p = 1200$, Log $\Gamma(p)$ is given for every integral value of p; the values of the function have been taken from Degen's Table of the Logarithms of factorials†, where of course Log $(p\,!) = $ Log $\Gamma(p+1)$. The differences were worked out using twelve decimal places, and the last two figures then cut off, but here again when doubtful cases arose (i.e. when the 11th figure was nearly 5) it was necessary to examine the 13th or 14th figures.

The Table may of course be used to find values of the Logarithms of Factorials. To obtain from it values of Log $\Gamma(p)$, interpolation will in general be necessary, and the differences tabled are the second, fourth and sixth differences—δ^2, δ^4 and δ^6—which should be used in Everett's Central Difference Interpolation Formula‡:

$$\begin{aligned} z_\theta = \phi z_0 + \theta z_1 &- \tfrac{1}{6}\phi\theta\{(1+\phi)\,\delta^2 z_0 + (1+\theta)\,\delta^2 z_1\} \\ &+ \tfrac{1}{120}\phi\theta(1+\phi)(1+\theta)\{(2+\phi)\,\delta^4 z_0 + (2+\theta)\,\delta^4 z_1\} \\ &- \tfrac{1}{5040}\phi\theta(1+\phi)(1+\theta)(2+\phi)(2+\theta)\{(3+\phi)\,\delta^6 z_0 + (3+\theta)\,\delta^6 z_1\} + \ldots, \end{aligned}$$

where $\qquad z_0 = f(x_0), \qquad z_1 = f(x_0 + h), \qquad z_\theta = f(x_0 + \theta h),$

h is the argument interval and θ and ϕ are two positive quantities such that $\theta + \phi = 1$.

If this formula be written

$$\begin{aligned} z_\theta = \phi z_0 + \theta z_1 - [\epsilon_2(\phi)\,\delta^2 z_0 + \epsilon_2(\theta)\,\delta^2 z_1] + [\epsilon_4(\phi)\,\delta^4 z_0 + \epsilon_4(\theta)\,\delta^4 z_1] \\ - [\epsilon_6(\phi)\,\delta^6 z_0 + \epsilon_6(\theta)\,\delta^6 z_1] + \ldots, \end{aligned}$$

* A number of last (i.e. 12th) figure errors have been found in Legendre's Table. In testing his values against those of Gauss (who goes to twenty figures), an error of a unit was found in twenty-three cases out of a hundred; the errors were both positive and negative.

† See footnote † to p. iv above.

‡ *Journal of the Institute of Actuaries*, Vol. xxxv. p. 452.

the functions ϵ_2, ϵ_4 and ϵ_6 have been tabled for values of θ (and therefore of ϕ) lying between zero and unity by Mr A. J. Thompson in Tract No. V of this Series*. Using his Table the whole process of an interpolation can be carried out by a practised computer in one continuous operation on the machine. The functions are tabled to ten figures; if the value of Log $\Gamma(p)$ be required to the full ten places of decimals, it is advisable to take out ϵ_2 to eight figures, while for seven place accuracy in Log $\Gamma(p)$, only five figures of ϵ_2 need be used. For ϵ_4 less figures are actually needed, but if the operation is to be a continuous one on the machine, little is gained and mistakes may be introduced by taking out fewer figures than for ϵ_2.

It may be stated as a rough guide, that when working with Everett's Formula it is unnecessary to use a 2nd difference under 4, a 4th difference under 20, or a 6th difference under 100 (where these differences correspond to the last figures of the tabled entries). Thus in working to ten decimal places, the 6th differences of Log $\Gamma(p)$ are only required within the range $p = 2$ to $p = 10$, while 4th differences become negligible at about $p = 800$. If working to seven decimal places only, 6th differences will not be required at all, and 4th differences only between $p = 2$ and $p = 10$, and again from $p = 70$ to $p = 80$. These rules will serve for practical purposes, although an error of a unit in the last place may occur in unfavourable circumstances; in fact it is obvious that in any process involving the addition of quantities cut off at a certain figure, this possibility cannot be eliminated altogether although the chance of the occurrence of a unit last figure error may be reduced by retaining differences below the limits given above.

III. *Illustrative Examples.*

Example 1.

To find Log $\Gamma(6{\cdot}8234)$ to ten places of decimals. At this point of the Table the argument interval is $0{\cdot}2$; we find

$$z_0 = \log\Gamma(6{\cdot}8) = 2{\cdot}696\ 012\ 0302, \qquad z_1 = \log\Gamma(7{\cdot}0) = 2{\cdot}857\ 332\ 4964,$$
$$\delta^2 z_0 = 2\ 752\ 1435, \qquad \delta^2 z_1 = 2\ 667\ 7699,$$
$$\delta^4 z_0 = 5\ 5003, \qquad \delta^4 z_1 = 5\ 0111,$$
$$\delta^6 z_0 = 657, \qquad \delta^6 z_1 = 562.$$

$$\theta = \frac{{\cdot}234}{2} = {\cdot}117, \qquad \phi = {\cdot}883.$$

$$\epsilon_2(\phi) = {\cdot}03242\ 244, \qquad \epsilon_2(\theta) = {\cdot}01923\ 306,$$
$$\epsilon_4(\phi) = 522\ 052, \qquad \epsilon_4(\theta) = 383\ 345,$$
$$\epsilon_6(\phi) = 102\ 177, \qquad \epsilon_6(\theta) = 82\ 020.$$

* *Tracts for Computers*, No. V. Cambridge University Press.

Hence we find, taking the terms separately,

$$\begin{aligned}
\phi z_0 + \theta z_1 &= +\,2{\cdot}714\ 886\ 5247(5)\\
-[\epsilon_2(\phi)\,\delta^2 z_0 + \epsilon_2(\theta)\,\delta^2 z_1] &= -\qquad 140\ 5405(9)\\
+[\epsilon_4(\phi)\,\delta^4 z_0 + \epsilon_4(\theta)\,\delta^4 z_1] &= +\qquad 479(2)\\
-[\epsilon_6(\phi)\,\delta^6 z_0 + \epsilon_6(\theta)\,\delta^6 z_1] &= -\qquad 1(1)\\
\operatorname{Log}\Gamma(6{\cdot}8234) &= \underline{2{\cdot}714\ 746\ 0320}
\end{aligned}$$

By the addition of the logarithms of 1·8234, 2·8234, 3·8234, 4·8234 and 5·8234 to the value of log Γ (1·8234) interpolated from Legendre's Table, the same result was obtained*. Were Stirling's asymptotic expansion used, it would be necessary to include the term $\frac{1}{1188p^9}$ to be certain of tenth figure accuracy.

Example 2.

To find Log Γ (691·43) to ten places of decimals.

$z_0 = \log\Gamma(691) = 1660{\cdot}961\ 247\ 0260$, $z_1 = \log\Gamma(692) = 1663{\cdot}800\ 725\ 0734$,
$\delta^2 z_0 = 628\ 9566$, $\delta^2 z_1 = 628\ 0471$,
$\delta^4 z_0 = 26$, $\delta^4 z_1 = 26$.

$$\phi = {\cdot}57, \qquad \theta = {\cdot}43.$$

$$\begin{aligned}
\phi z_0 + \theta z_1 &= 1662{\cdot}182\ 222\ 5863(8)\\
-[\epsilon_2(\phi)\,\delta^2 z_0 + \epsilon_2(\theta)\,\delta^2 z_1] &= -\qquad 77\ 0255(0)\\
+[\epsilon_4(\phi)\,\delta^4 z_0 + \epsilon_4(\theta)\,\delta^4 z_1] &= \qquad (6)\\
\operatorname{Log}\Gamma(691{\cdot}43) &= \underline{1662{\cdot}182\ 145\ 5609}
\end{aligned}$$

Example 3.

If θ contains more than three decimal places it becomes necessary to interpolate into Thompson's Table to obtain the values of the ϵ functions. This of course lengthens the process of computation, but if Log Γ (p) is required to seven decimal places only, ϵ_2 need only be taken to five places and linear interpolation for its value is sufficient throughout the Table. The process is illustrated in the following example.

To find Log Γ (72·38365) to seven places of decimals.

$z_0 = \operatorname{Log}\Gamma(72) = 101{\cdot}929\ 6634$, $z_1 = \log\Gamma(73) = 103{\cdot}786\ 9959$,
$\delta^2 z_0 = 6\ 0741$, $\delta^2 z_1 = 5\ 9904$,
$\delta^4 z_0 = 24$, $\delta^4 z_1 = 23$.

* The process is however lengthy, since for last figure accuracy it is necessary to work with logarithms to eleven places, and even using Briggs' Table interpolation may be necessary.

$$\phi = \cdot 61635, \qquad \theta = \cdot 38365.$$

$$\begin{array}{ll} \epsilon_2(616) = \cdot 063709 & \epsilon_2(\cdot 383) = \cdot 054470 \\ \epsilon_2(617) = \cdot 063686 & \epsilon_2(\cdot 384) = \cdot 054563 \\ \Delta = - \quad 23 & \Delta = + \quad 93 \end{array}$$

Hence
$$\begin{array}{ll} \epsilon_2(\phi) = \cdot 06370, & \epsilon_2(\theta) = \cdot 05453, \\ \epsilon_4(\phi) = \cdot 01153 \text{ (at sight)}, & \epsilon_4(\theta) = \cdot 01050 \text{ (at sight)}. \end{array}$$

$$\begin{aligned} \phi z_0 + \theta z_1 &= 102\cdot 642\ 2290(1) \\ -[\epsilon_2(\phi)\,\delta^2 z_0 + \epsilon_2(\theta)\,\delta^2 z_1] &= - \qquad\quad 7135(8) \\ [\epsilon_4(\phi)\,\delta^4 z_0 + \epsilon_4(\theta)\,\delta^4 z_1] &= \qquad\qquad\quad (5) \\ \operatorname{Log}\Gamma(72\cdot 38365) &= \underline{102\cdot 641\ 5155} \end{aligned}$$

IV. *The Derivatives of Log* $\Gamma(p)$.

It is also possible to use the present Table to calculate the values of the successive Derivatives of the Logarithm of the Gamma Function, or of

$$\digamma(p) = \frac{d}{dp}\log_e \Gamma(1+p) = -\gamma + \sum_{n=1}^{\infty} \frac{p}{n(n+p)},$$

where
$$\gamma = 0\cdot 5772157\ldots, \text{ or Euler's Constant.}$$

$$\digamma\!\!\digamma(p) = \frac{d^2}{dp^2}\log_e \Gamma(1+p) = \sum_{n=1}^{\infty} \frac{1}{(n+p)^2},$$

$$\frac{d^3}{dp^3}\log_e \Gamma(1+p) = -2\sum_{n=1}^{\infty} \frac{1}{(n+p)^3}, \text{ etc.}$$

Values of the first function, $\frac{d}{dp}\log_e \Gamma(1+p)$, have been given by Gauss* to eighteen decimal places for the range $p=0$ to $p=1$ with argument interval ·01, and also by Prof. G. N. Watson† to thirteen decimal places for all integers and halves of odd integers from 0 to 100. *Tracts for Computers*, No. 1‡, contains Tables to eight decimal places of both $\digamma(p)$ and $\digamma\!\!\digamma(p)$, for the range $p=0$ to $p=20$, and with argument interval ·02.

The value of all these functions can however be obtained by interpolation from the present Table, with the help of the following central difference interpolation formulae for the derivatives of $z_\theta = f(x_0 + \theta h)$.

* *Werke*, Bd III. S. 161, 162. Göttingen, 1866.

† *Report of British Association*, 1916, pp. 125 and 126.

‡ Cambridge University Press. The symbols $\digamma$ and $\digamma\!\!\digamma$ and the names Digamma and Trigamma Functions were there first suggested. $\frac{d^3}{dp^3}\log_e \Gamma(1+p)$ might be called the Tetragamma Function.

$$\text{(i)}\quad h\frac{dz_\theta}{dx}=\phi\frac{z_1-z_{-1}}{2}+\theta\frac{z_2-z_0}{2}$$
$$-\left[\{\tfrac{1}{6}\phi+\epsilon_2(\phi)\}\frac{\delta^2 z_1-\delta^2 z_{-1}}{2}+\{\tfrac{1}{6}\theta+\epsilon_2(\theta)\}\frac{\delta^2 z_2-\delta^2 z_0}{2}\right]$$
$$+\left[\{\tfrac{1}{30}\phi+\tfrac{1}{6}\epsilon_2(\phi)+\epsilon_4(\phi)\}\frac{\delta^4 z_1-\delta^4 z_{-1}}{2}+\{\tfrac{1}{30}\theta+\tfrac{1}{6}\epsilon_2(\theta)+\epsilon_4(\theta)\}\frac{\delta^4 z_2-\delta^4 z_0}{2}\right]$$
$$-\ldots\ldots,$$

$$\text{(ii)}\quad h^2\frac{d^2z}{dx^2}=\phi\delta^2 z_0+\theta\delta^2 z_1-[\{\tfrac{1}{12}\phi+\epsilon_2(\phi)\}\,\delta^4 z_0+\{\tfrac{1}{12}\theta+\epsilon_2(\theta)\}\,\delta^4 z_1]$$
$$+[\{\tfrac{1}{90}\phi+\tfrac{1}{12}\epsilon_2(\phi)+\epsilon_4(\phi)\}\,\delta^6 z_0+\{\tfrac{1}{90}\theta+\tfrac{1}{12}\epsilon_2(\theta)+\epsilon_4(\theta)\}\,\delta^6 z_1]$$
$$-\ldots\ldots,$$

$$\text{(iii)}\quad h^3\frac{d^3z}{dx^3}=\phi\frac{\delta^2 z_1-\delta^2 z_{-1}}{2}+\theta\frac{\delta^2 z_2-\delta^2 z_0}{2}$$
$$-\left[\{\tfrac{1}{4}\phi+\epsilon_2(\phi)\}\frac{\delta^4 z_1-\delta^4 z_{-1}}{2}+\{\tfrac{1}{4}\theta+\epsilon_2(\theta)\}\frac{\delta^4 z_2-\delta^4 z_0}{2}\right]$$
$$+\left[\{\tfrac{7}{120}\phi+\tfrac{1}{4}\epsilon_2(\phi)+\epsilon_4(\phi)\}\frac{\delta^6 z_1-\delta^6 z_{-1}}{2}+\{\tfrac{7}{120}\theta+\tfrac{1}{4}\epsilon_2(\theta)+\epsilon_4(\theta)\}\frac{\delta^6 z_2-\delta^6 z_0}{2}\right]$$
$$-\ldots\ldots,$$

where h is the argument interval, θ, ϕ, ϵ_2, and ϵ_4 are the Everett coefficients referred to above, and

$$z_{-1}=f(x_0-h);\; z_0=f(x_0);\; z_1=f(x_0+h);\; z_2=f(x_0+2h).$$

The following rules provide a rough guide to the order of differences which can be neglected:

In (i) it is unnecessary to use the term containing the 4th differences if 5th differences are less than 6;

in (ii) it is unnecessary to use 6th differences less than 10;

in (iii) the term containing 6th differences is negligible if 7th differences are less than 4;

where the differences correspond to the last figures of the tabled entries.

It will be noted that if $\theta=0$ and $\phi=1$, or we require the value of a derivative of the function for a value of x at which the function is tabled, the equations (i), (ii) and (iii) reduce to much simpler forms, for in this case

$$\epsilon_2(\phi)=\epsilon_2(\theta)=0,\qquad \epsilon_4(\phi)=\epsilon_4(\theta)=0.$$

Example.

To find $$\digamma(37{\cdot}68) = \frac{d^2}{dp^2} \log_{10} \Gamma(38{\cdot}68) \times \log_e 10.$$

Here $$z_0 = \log \Gamma(38{\cdot}6), \qquad z_1 = \log \Gamma(38{\cdot}8),$$
$$\delta^2 z_0 = {\cdot}000\ 455\ 9281, \qquad \delta^2 z_1 = {\cdot}000\ 453\ 5476,$$
$$\delta^4 z_0 = \qquad 251, \qquad \delta^4 z_1 = \qquad 247.$$

The sixth difference will be less than 10 and is therefore negligible.

$$h = {\cdot}2, \qquad \phi = {\cdot}6, \qquad \theta = {\cdot}4.$$

$$\begin{array}{ll} \epsilon_2(\phi) = {\cdot}06400\ 00 & \epsilon_2(\theta) = {\cdot}05600\ 00 \\ \tfrac{1}{12}\phi \ = {\cdot}05000\ 00 & \tfrac{1}{12}\theta \ = {\cdot}03333\ 33 \\ \hline \qquad {\cdot}11400\ 00 & \qquad {\cdot}08933\ 33 \end{array}$$

$$\begin{array}{rr} \phi\delta^2 z_0 + \theta\delta^2 z_1 = & {\cdot}000\ 454\ 9759(0) \\ -\left[\{\tfrac{1}{12}\phi + \epsilon_2(\phi)\}\,\delta^4 z_0 + \{\tfrac{1}{12}\theta + \epsilon_2(\theta)\}\,\delta^4 z_1\right] = - & 50(7) \\ \hline & {\cdot}000\ 454\ 9708(3) \end{array}$$

Multiplying ·000 454 97083 by $\log_e 10$ (= 2·302 585 0930) and dividing by h^2 (= ·04) we have finally

$$\digamma(37{\cdot}68) = {\cdot}0261\ 9023 \text{ to eight decimal places};$$

this result agrees with that given by Miss Pairman as an example on p. 6 of the introduction to her Table*.

* See footnote to p. viii above.

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TABLE OF THE LOGARITHMS OF THE COMPLETE Γ-FUNCTION, ARGUMENTS 2 TO 1200

p	Log $\Gamma(p)$	δ^2	δ^4	δ^6	p
2·0	0·000 000 0000	2 802 7032	21 6033	9266	2·0
2·1	0·019 733 3591	2 637 0316	18 1288	6976	2·1
2·2	0·042 103 7499	2 489 4889	15 3520	5327	2·2
2·3	0·066 963 6295	2 357 2980	13 1078	4121	2·3
2·4	0·094 180 8071	2 238 2150	11 2757	3225	2·4
2·5	0·123 636 1997	2 130 4077	9 7662	2552	2·5
2·6	0·155 222 0001	2 032 3666	8 5118	2039	2·6
2·7	0·188 840 1671	1 942 8373	7 4614	1644	2·7
2·8	0·224 401 1713	1 860 7694	6 5754	1337	2·8
2·9	0·261 822 9450	1 785 2769	5 8231	1096	2·9
3·0	0·301 029 9957	1 715 6076	5 1804	905	3·0
3·1	0·341 952 6539	1 651 1186	4 6282	752	3·1
3·2	0·384 526 4307	1 591 2580	4 1513	629	3·2
3·3	0·428 691 4655	1 535 5485	3 7372	529	3·3
3·4	0·474 392 0488	1 483 5763	3 3760	448	3·4
3·5	0·521 576 2084	1 434 9801	3 0597	381	3·5
3·6	0·570 195 3481	1 389 4435	2 7814	326	3·6
3·7	0·620 203 9312	1 346 6883	2 5357	280	3·7
3·8	0·671 559 2027	1 306 4688	2 3179	241	3·8
3·9	0·724 220 9429	1 268 5672	2 1243	209	3·9
4·0	0·778 151 2504	1 232 7899	1 9515	182	4·0
4·1	0·833 314 3477	1 198 9640	1 7969	158	4·1
4·2	0·889 676 4090	1 166 9350	1 6581	139	4·2
4·3	0·947 205 4054	1 136 5641	1 5332	122	4·3
4·4	1·005 870 9658	1 107 7264	1 4205	107	4·4
4·5	1·065 644 2528	1 080 3092	1 3185	95	4·5
4·6	1·126 497 8488	1 054 2104	1 2260	84	4·6
4·7	1·188 405 6553	1 029 3376	1 1419	75	4·7
4·8	1·251 342 7993	1 005 6067	1 0653	67	4·8
4·9	1·315 285 5500	982 9411	9954	60	4·9
5·0	1·380 211 2417	961 2709	9315	53	5·0
5·0	1·380 211 2417	3 846 0153	14 9466	3457	5·0
5·2	1·512 925 6994	3 683 4852	13 1401	2791	5·2
5·4	1·649 323 6423	3 534 0953	11 6127	2273	5·4
5·6	1·789 255 6805	3 396 3180	10 3126	1867	5·6
5·8	1·932 584 0367	3 268 8532	9 1991	1544	5·8
6·0	2·079 181 2460	3 150 5876	8 2401	1286	6·0
6·2	2·228 929 0430	3 040 5621	7 4096	1078	6·2
6·4	2·381 717 4022	2 937 9463	6 6869	909	6·4
6·6	2·537 443 7075	2 842 0173	6 0551	770	6·6
6·8	2·696 012 0302	2 752 1435	5 5003	657	6·8
7·0	2·857 332 4964	2 667 7699	5 0111	562	7·0
7·2	3·021 320 7325	2 588 4075	4 5783	484	7·2
7·4	3·187 897 3761	2 513 6233	4 1938	418	7·4
7·6	3·356 987 6431	2 443 0329	3 8511	363	7·6
7·8	3·528 520 9429	2 376 2936	3 5447	316	7·8
8·0	3·702 430 5364	2 313 0990	3 2700	276	8·0
8·2	3·878 663 2290	2 253 1744	3 0228	243	8·2
8·4	4·057 129 0959	2 196 2726	2 8000	214	8·4
8·6	4·237 801 2354	2 142 1708	2 5985	189	8·6
8·8	4·420 615 5456	2 090 6675	2 4159	167	8·8
9·0	4·605 520 5234	2 041 5801	2 2500	148	9·0
9·2	4·792 467 0813	1 994 7427	2 0989	132	9·2
9·4	4·981 408 3819	1 950 0041	1 9610	118	9·4
9·6	5·172 299 6866	1 907 2265	1 8349	105	9·6
9·8	5·365 098 2178	1 866 2839	1 7195	95	9·8
10·0	5·559 763 0329	1 827 0607	1 6135	85	10·0

p	$\operatorname{Log}\Gamma(p)$	δ^2	δ^4
10·0	5·559 763 0329	1 827 0607	1 6135
10·2	5·756 254 9087	1 789 4510	1 5160
10·4	5·954 536 2355	1 753 3573	1 4262
10·6	6·154 570 9196	1 718 6897	1 3434
10·8	6·356 324 2935	1 685 3656	1 2668
11·0	6·559 763 0329	1 653 3082	1 1960
11·2	6·764 855 0805	1 622 4468	1 1303
11·4	6·971 569 5748	1 592 7157	1 0694
11·6	7·179 876 7849	1 564 0540	1 0127
11·8	7·389 748 0490	1 536 4050	9600
12·0	7·601 155 7180	1 509 7160	9109
12·2	7·814 073 1031	1 483 9379	8651
12·4	8·028 474 4262	1 459 0249	8223
12·6	8·244 334 7741	1 434 9342	7822
12·8	8·461 630 0563	1 411 6257	7448
13·0	8·680 336 9641	1 389 0619	7096
13·2	8·900 432 9338	1 367 2078	6767
13·4	9·121 896 1113	1 346 0304	6458
13·6	9·344 705 3192	1 325 4988	6167
13·8	9·568 840 0259	1 305 5838	5893
14·0	9·794 280 3164	1 286 2582	5636
14·2	10·021 006 8650	1 267 4961	5393
14·4	10·249 000 9097	1 249 2732	5164
14·6	10·478 244 2276	1 231 5668	4947
14·8	10·708 719 1123	1 214 3550	4743
15·0	10·940 408 3521	1 197 6176	4550
15·2	11·173 295 2094	1 181 3351	4367
15·4	11·407 363 4018	1 165 4892	4193
15·6	11·642 597 0834	1 150 0627	4029
15·8	11·878 980 8277	1 135 0391	3873
16·0	12·116 499 6111	1 120 4028	3725
16·2	12·355 138 7973	1 106 1391	3585
16·4	12·594 884 1226	1 092 2338	3452
16·6	12·835 721 6818	1 078 6738	3325
16·8	13·077 637 9147	1 065 4462	3204
17·0	13·320 619 5938	1 052 5390	3089
17·2	13·564 653 8119	1 039 9407	2979
17·4	13·809 727 9707	1 027 6403	2875
17·6	14·055 829 7698	1 015 6275	2775
17·8	14·302 947 1964	1 003 8922	2680
18·0	14·551 068 5152	992 4249	2590
18·2	14·800 182 2588	981 2165	2503
18·4	15·050 277 2190	970 2585	2420
18·6	15·301 342 4376	959 5424	2341
18·8	15·553 367 1987	949 0605	2265
19·0	15·806 341 0203	938 8050	2192
19·2	16·060 253 6468	928 7687	2123
19·4	16·315 095 0420	918 9447	2056
19·6	16·570 855 3818	909 3263	1992
19·8	16·827 525 0480	899 9071	1931
20·0	17·085 094 6212	890 6810	1872

p	$\operatorname{Log}\Gamma(p)$	δ^2	δ^4
20·0	17·085 094 6212	890 6810	187
20·2	17·343 554 8755	881 6422	181
20·4	17·602 896 7719	872 7849	176
20·6	17·863 111 4532	864 1038	171
20·8	18·124 190 2382	855 5936	166
21·0	18·386 124 6169	847 2494	161
21·2	18·648 906 2449	839 0664	156
21·4	18·912 526 9393	831 0398	152
21·6	19·176 978 6736	823 1654	147
21·8	19·442 253 5732	815 4388	143
22·0	19·708 343 9116	807 8558	139
22·2	19·975 242 1058	800 4126	135
22·4	20·242 940 7127	793 1052	132
22·6	20·511 432 4247	785 9301	128
22·8	20·780 710 0668	778 8835	125
23·0	21·050 766 5924	771 9622	121
23·2	21·321 595 0803	765 1628	118
23·4	21·593 188 7310	758 4821	115
23·6	21·865 540 8639	751 9171	112
23·8	22·138 644 9138	745 4647	109
24·0	22·412 494 4285	739 1221	107
24·2	22·687 083 0652	732 8865	1043
24·4	22·962 404 5884	726 7552	1017
24·6	23·238 452 8668	720 7256	992
24·8	23·515 221 8709	714 7953	96
25·0	23·792 705 6702	708 9617	94
25·2	24·070 898 4312	703 2226	92
25·4	24·349 794 4148	697 5756	90
25·6	24·629 387 9739	692 0186	87
25·8	24·909 673 5517	686 5494	85
26·0	25·190 645 6788	681 1660	83
26·2	25·472 298 9720	675 8663	81
26·4	25·754 628 1314	670 6485	79
26·6	26·037 627 9392	665 5105	78
26·8	26·321 293 2576	660 4508	76
27·0	26·605 619 0268	655 4673	74
27·2	26·890 600 2633	650 5585	73
27·4	27·176 232 0582	645 7227	71
27·6	27·462 509 5759	640 9582	69
27·8	27·749 428 0517	636 2635	68
28·0	28·036 982 7910	631 6371	66
28·2	28·325 169 1673	627 0774	65
28·4	28·613 982 6211	622 5831	64
28·6	28·903 418 6579	618 1528	62
28·8	29·193 472 8476	613 7851	61
29·0	29·484 140 8223	609 4786	60
29·2	29·775 418 2756	605 2322	58
29·4	30·067 300 9611	601 0445	57
29·6	30·359 784 6911	596 9143	56
29·8	30·652 865 3354	592 8406	55
30·0	30·946 538 8202	588 8220	54

p	$\operatorname{Log}\Gamma(p)$	δ^2	δ^4
30·0	30·946 538 8202	588 8220	541
30·2	31·240 801 1271	584 8576	530
30·4	31·535 648 2915	580 9462	520
30·6	31·831 076 4021	577 0867	509
30·8	32·127 081 5994	573 2782	499
31·0	32·423 660 0749	569 5196	490
31·2	32·720 808 0700	565 8100	480
31·4	33·018 521 8751	562 1484	471
31·6	33·316 797 8286	558 5339	462
31·8	33·615 632 3159	554 9655	453
32·0	33·915 021 7688	551 4425	444
32·2	34·214 962 6641	547 9639	436
32·4	34·515 451 5232	544 5289	428
32·6	34·816 484 9112	541 1367	420
32·8	35·118 059 4359	537 7865	412
33·0	35·420 171 7471	534 4775	405
33·2	35·722 818 5357	531 2090	397
33·4	36·025 996 5334	527 9802	390
33·6	36·329 702 5113	524 7905	383
33·8	36·633 933 2796	521 6390	376
34·0	36·938 685 6870	518 5252	370
34·2	37·243 956 6195	515 4483	363
34·4	37·549 743 0002	512 4077	357
34·6	37·856 041 7887	509 4028	350
34·8	38·162 849 9799	506 4329	344
35·0	38·470 164 6040	503 4974	338
35·2	38·777 982 7255	500 5958	333
35·4	39·086 301 4428	497 7274	327
35·6	39·395 117 8875	494 8917	321
35·8	39·704 429 2238	492 0881	316
36·0	40·014 232 6484	489 3161	311
36·2	40·324 525 3890	486 5752	305
36·4	40·635 304 7048	483 8648	300
36·6	40·946 567 8854	481 1844	295
36·8	41·258 312 2505	478 5336	290
37·0	41·570 535 1491	475 9118	286
37·2	41·883 233 9595	473 3186	281
37·4	42·196 406 0885	470 7534	277
37·6	42·510 048 9708	468 2160	272
37·8	42·824 160 0692	465 7057	268
38·0	43·138 736 8732	463 2222	263
38·2	43·453 776 8994	460 7651	259
38·4	43·769 277 6907	458 3338	255
38·6	44·085 236 8158	455 9281	251
38·8	44·401 651 8690	453 5476	247
39·0	44·718 520 4698	451 1917	243
39·2	45·035 840 2623	448 8602	240
39·4	45·353 608 9150	446 5527	236
39·6	45·671 824 1204	444 2688	232
39·8	45·990 483 5946	442 0081	229
40·0	46·309 585 0768	439 7703	225

p	$\operatorname{Log}\Gamma(p)$	δ^2	δ^4
40·0	46·309 585 0768	439 7703	225
40·2	46·629 126 3293	437 5550	222
40·4	46·949 105 1369	435 3620	219
40·6	47·269 519 3064	433 1908	215
40·8	47·590 366 6667	431 0412	212
41·0	47·911 645 0682	428 9128	209
41·2	48·233 352 3824	426 8053	206
41·4	48·555 486 5020	424 7184	203
41·6	48·878 045 3399	422 6518	200
41·8	49·201 026 8298	420 6053	197
42·0	49·524 428 9249	418 5785	194
42·2	49·848 249 5984	416 5711	192
42·4	50·172 486 8431	414 5828	189
42·6	50·497 138 6706	412 6135	186
42·8	50·822 203 1115	410 6628	184
43·0	51·147 678 2153	408 7304	181
43·2	51·473 562 0494	406 8161	178
43·4	51·799 852 6997	404 9197	176
43·6	52·126 548 2697	403 0409	173
43·8	52·453 646 8806	401 1794	171
44·0	52·781 146 6709	399 3351	169
44·2	53·109 045 7962	397 5076	166
44·4	53·437 342 4292	395 6968	164
44·6	53·766 034 7589	393 9024	162
44·8	54·095 120 9911	392 1242	160
45·0	54·424 599 3473	390 3620	158
45·2	54·754 468 0656	388 6155	156
45·4	55·084 725 3993	386 8846	153
45·6	55·415 369 6177	385 1691	151
45·8	55·746 399 0051	383 4687	149
46·0	56·077 811 8611	381 7832	148
46·2	56·409 606 5004	380 1125	146
46·4	56·741 781 2522	378 4564	144
46·6	57·074 334 4603	376 8146	142
46·8	57·407 264 4831	375 1870	140
47·0	57·740 569 6928	373 5734	138
47·2	58·074 248 4759	371 9736	136
47·4	58·408 299 2327	370 3875	135
47·6	58·742 720 3770	368 8148	133
47·8	59·077 510 3361	367 2555	131
48·0	59·412 667 5507	365 7092	130
48·2	59·748 190 4746	364 1760	128
48·4	60·084 077 5744	362 6555	126
48·6	60·420 327 3297	361 1477	125
48·8	60·756 938 2327	359 6523	123
49·0	61·093 908 7881	358 1693	122
49·2	61·431 237 5128	356 6985	120
49·4	61·768 922 9360	355 2397	119
49·6	62·106 963 5990	353 7928	117
49·8	62·445 358 0547	352 3576	116
50·0	62·784 104 8681	350 9341	115

p	$\operatorname{Log}\Gamma(p)$	δ^2	δ^4
50·0	62·784 104 8681	350 9341	115
50·2	63·123 202 6156	349 5219	113
50·4	63·462 649 8850	348 1211	112
50·6	63·802 445 2755	346 7315	110
50·8	64·142 587 3975	345 3529	109
51·0	64·483 074 8725	343 9853	108
51·2	64·823 906 3327	342 6284	107
51·4	65·165 080 4214	341 2822	105
51·6	65·506 595 7923	339 9466	104
51·8	65·848 451 1098	338 6213	103
52·0	66·190 645 0486	337 3064	102
52·2	66·533 176 2937	336 0016	101
52·4	66·876 043 5404	334 7068	99
52·6	67·219 245 4939	333 4220	98
52·8	67·562 780 8695	332 1471	97
53·0	67·906 648 3922	330 8818	96
53·2	68·250 846 7967	329 6262	95
53·4	68·595 374 8274	328 3800	94
53·6	68·940 231 2381	327 1433	93
53·8	69·285 414 7921	325 9158	92
54·0	69·630 924 2618	324 6975	91
54·2	69·976 758 4290	323 4882	90
54·4	70·322 916 0844	322 2880	89
54·6	70·669 396 0278	321 0966	88
54·8	71·016 197 0677	319 9140	87
55·0	71·363 318 0216	318 7400	86
55·2	71·710 757 7155	317 5747	85
55·4	72·058 514 9841	316 4178	84
55·6	72·406 588 6705	315 2693	83
55·8	72·754 977 6262	314 1292	82
56·0	73·103 680 7111	312 9972	81
56·2	73·452 696 7933	311 8734	80
56·4	73·802 024 7488	310 7577	80
56·6	74·151 663 4621	309 6498	79
56·8	74·501 611 8252	308 5499	78
57·0	74·851 868 7381	307 4577	77
57·2	75·202 433 1088	306 3733	76
57·4	75·553 303 8528	305 2964	75
57·6	75·904 479 8933	304 2272	75
57·8	76·255 960 1609	303 1653	74
58·0	76·607 743 5938	302 1109	73
58·2	76·959 829 1376	301 0638	72
58·4	77·312 215 7452	300 0239	72
58·6	77·664 902 3767	298 9911	71
58·8	78·017 887 9993	297 9655	70
59·0	78·371 171 5874	296 9468	69
59·2	78·724 752 1223	295 9351	69
59·4	79·078 628 5923	294 9303	68
59·6	79·432 799 9927	293 9323	67
59·8	79·787 265 3254	292 9410	67
60·0	80·142 023 5990	291 9564	66

p	$\operatorname{Log}\Gamma(p)$	δ^2	δ^4
60·0	80·142 023 5990	291 9564	66
60·2	80·497 073 8290	290 9783	65
60·4	80·852 415 0373	290 0068	65
60·6	81·208 046 2524	289 0418	64
60·8	81·563 966 5094	288 0831	63
61·0	81·920 174 8494	287 1308	63
61·2	82·276 670 3203	286 1848	62
61·4	82·633 451 9759	285 2450	62
61·6	82·990 518 8766	284 3113	61
61·8	83·347 870 0886	283 3838	60
62·0	83·705 504 6844	282 4622	60
62·2	84·063 421 7424	281 5467	59
62·4	84·421 620 3471	280 6370	59
62·6	84·780 099 5888	279 7332	58
62·8	85·138 858 5637	278 8353	57
63·0	85·497 896 3739	277 9430	57
63·2	85·857 212 1271	277 0565	56
63·4	86·216 804 9368	276 1756	56
63·6	86·576 673 9220	275 3002	55
63·8	86·936 818 2075	274 4304	55
64·0	87·297 236 9234	273 5661	54
64·2	87·657 929 2054	272 7072	54
64·4	88·018 894 1946	271 8537	53
64·6	88·380 131 0376	271 0055	53
64·8	88·741 638 8862	270 1626	52
65·0	89·103 416 8973	269 3249	52
65·2	89·465 464 2334	268 4924	51
65·4	89·827 780 0620	267 6651	51
65·6	90·190 363 5556	266 8428	50
65·8	90·553 213 8920	266 0255	50
66·0	90·916 330 2540	265 2133	49
66·2	91·279 711 8292	264 4059	49
66·4	91·643 357 8103	263 6035	49
66·6	92·007 267 3950	262 8060	48
66·8	92·371 439 7857	262 0132	48
67·0	92·735 874 1895	261 2252	47
67·2	93·100 569 8186	260 4420	47
67·4	93·465 525 8897	259 6634	46
67·6	93·830 741 6242	258 8895	46
67·8	94·196 216 2481	258 1201	46
68·0	94·561 948 9922	257 3554	45
68·2	94·927 939 0917	256 5951	45
68·4	95·294 185 7862	255 8393	44
68·6	95·660 688 3201	255 0880	44
68·8	96·027 445 9420	254 3411	44
69·0	96·394 457 9049	253 5985	43
69·2	96·761 723 4663	252 8602	43
69·4	97·129 241 8880	252 1263	42
69·6	97·497 012 4358	251 3965	42
69·8	97·865 034 3802	250 6710	42
70·0	98·233 306 9957	249 9497	41

p	$\operatorname{Log}\Gamma(p)$	δ^2	δ^4
70	98·233 306 9957	6 248 9493	2 5882
71	100·078 405 0357	6 160 3087	2 4796
72	101·929 663 3844	6 074 1477	2 3770
73	103·786 995 8808	5 990 3637	2 2799
74	105·650 318 7410	5 908 8596	2 1881
75	107·519 550 4607	5 829 5437	2 1012
76	109·394 611 7241	5 752 3289	2 0188
77	111·275 425 3164	5 677 1329	1 9406
78	113·161 916 0415	5 603 8775	1 8665
79	115·054 010 6442	5 532 4886	1 7960
80	116·951 637 7355	5 462 8957	1 7291
81	118·854 727 7225	5 395 0319	1 6654
82	120·763 212 7414	5 328 8335	1 6049
83	122·677 026 5938	5 264 2400	1 5472
84	124·596 104 6861	5 201 1937	1 4923
85	126·520 383 9722	5 139 6397	1 4399
86	128·449 802 8979	5 079 5255	1 3900
87	130·384 301 3492	5 020 8014	1 3423
88	132·323 820 6018	4 963 4195	1 2968
89	134·268 303 2739	4 907 3345	1 2533
90	136·217 693 2806	4 852 5028	1 2118
91	138·171 935 7900	4 798 8829	1 1721
92	140·130 977 1823	4 746 4350	1 1340
93	142·094 765 0097	4 695 1212	1 0977
94	144·063 247 9582	4 644 9050	1 0628
95	146·036 375 8118	4 595 7517	1 0294
96	148·014 099 4171	4 547 6278	9974
97	149·996 370 6502	4 500 5012	9667
98	151·983 142 3844	4 454 3414	9373
99	153·974 368 4601	4 409 1189	9090
100	155·970 003 6547	4 364 8054	8819
101	157·970 003 6547	4 321 3738	8558
102	159·974 325 0285	4 278 7980	8308
103	161·982 925 2003	4 237 0529	8067
104	163·995 762 4250	4 196 1146	7835
105	166·012 795 7643	4 155 9598	7612
106	168·033 985 0633	4 116 5662	7398
107	170·059 290 9286	4 077 9124	7192
108	172·088 674 7063	4 039 9778	6993
109	174·122 098 4618	4 002 7425	6801
110	176·159 524 9597	3 966 1872	6616
111	178·200 917 6449	3 930 2936	6438
112	180·246 240 6237	3 895 0439	6267
113	182·295 458 6463	3 860 4208	6101
114	184·348 537 0898	3 826 4079	5941
115	186·405 441 9411	3 792 9890	5787
116	188·466 139 7815	3 760 1489	5638
117	190·530 597 7707	3 727 8725	5494
118	192·598 783 6325	3 696 1456	5355
119	194·670 665 6398	3 664 9541	5220
120	196·746 212 6012	3 634 2847	5090

p	$\operatorname{Log}\Gamma(p)$	δ^2	δ^4
120	196·746 212 6012	3 634 2847	5090
121	198·825 393 8472	3 604 1243	4965
122	200·908 179 2175	3 574 4604	4843
123	202·994 539 0482	3 545 2808	4726
124	205·084 444 1597	3 516 5737	4612
125	207·177 865 8448	3 488 3278	4501
126	209·274 775 8578	3 460 5321	4395
127	211·375 146 4029	3 433 1758	4291
128	213·478 950 1239	3 406 2487	4191
129	215·586 160 0935	3 379 7407	4094
130	217·696 749 8038	3 353 6420	4000
131	219·810 693 1561	3 327 9433	3909
132	221·927 964 4518	3 302 6356	3820
133	224·048 538 3830	3 277 7098	3734
134	226·172 390 0240	3 253 1574	3651
135	228·299 494 8223	3 228 9701	3570
136	230·429 828 5908	3 205 1399	3492
137	232·563 367 4992	3 181 6588	3415
138	234·700 088 0664	3 158 5192	3341
139	236·839 967 1528	3 135 7139	3270
140	238·982 981 9530	3 113 2354	3200
141	241·129 109 9887	3 091 0770	3132
142	243·278 329 1014	3 069 2317	3066
143	245·430 617 4457	3 047 6931	3002
144	247·585 953 4832	3 026 4546	2940
145	249·744 315 9753	3 005 5101	2879
146	251·905 683 9775	2 984 8535	2820
147	254·070 036 8333	2 964 4790	2763
148	256·237 354 1681	2 944 3806	2707
149	258·407 615 8835	2 924 5530	2653
150	260·580 802 1519	2 904 9906	2600
151	262·756 893 4109	2 885 6882	2548
152	264·935 870 3582	2 866 6407	2498
153	267·117 713 9462	2 847 8429	2449
154	269·302 405 3770	2 829 2900	2402
155	271·489 926 0978	2 810 9773	2355
156	273·680 257 7960	2 792 9002	2310
157	275·873 382 3943	2 775 0541	2266
158	278·069 282 0468	2 757 4345	2223
159	280·267 939 1337	2 740 0374	2181
160	282·469 336 2580	2 722 8583	2141
161	284·673 456 2407	2 705 8934	2101
162	286·880 282 1167	2 689 1385	2062
163	289·089 797 1313	2 672 5899	2024
164	291·301 984 7357	2 656 2436	1987
165	293·516 828 5837	2 640 0962	1951
166	295·734 312 5279	2 624 1438	1916
167	297·954 420 6160	2 608 3831	1882
168	300·177 137 0871	2 592 8106	1848
169	302·402 446 3688	2 577 4229	1816
170	304·630 333 0735	2 562 2168	1784

p	$\operatorname{Log}\Gamma(p)$	δ^2	δ^4
170	304·630 333 0735	2 562 2168	1784
171	306·860 781 9948	2 547 1890	1753
172	309·093 778 1052	2 532 3365	1722
173	311·329 306 5521	2 517 6562	1692
174	313·567 352 6553	2 503 1452	1663
175	315·807 901 9035	2 488 8004	1635
176	318·050 939 9522	2 474 6191	1607
177	320·296 452 6200	2 460 5985	1580
178	322·544 425 8864	2 446 7359	1553
179	324·794 845 8887	2 433 0287	1527
180	327·047 698 9197	2 419 4741	1502
181	329·302 971 4248	2 406 0698	1477
182	331·560 649 9997	2 392 8131	1453
183	333·820 721 3876	2 379 7017	1429
184	336·083 172 4774	2 366 7333	1406
185	338·347 990 3004	2 353 9054	1383
186	340·615 162 0288	2 341 2158	1361
187	342·884 674 9730	2 328 6623	1339
188	345·156 516 5795	2 316 2427	1318
189	347·430 674 4288	2 303 9549	1297
190	349·707 136 2330	2 291 7968	1276
191	351·985 889 8339	2 279 7663	1256
192	354·236 923 2012	2 267 8615	1237
193	356·550 224 4299	2 256 0803	1218
194	358·835 781 7389	2 244 4209	1199
195	361·123 583 4688	2 232 8814	1181
196	363·413 618 0802	2 221 4600	1162
197	365·705 874 1515	2 210 1548	1145
198	368·000 340 3777	2 198 9641	1128
199	370·297 005 5680	2 187 8861	1111
200	372·595 858 6444	2 176 9193	1094
201	374·896 888 6400	2 166 0618	1078
202	377·200 084 6975	2 155 3120	1062
203	379·505 436 0669	2 144 6685	1046
204	381·812 932 1048	2 134 1295	1031
205	384·122 562 2722	2 123 6936	1016
206	386·434 316 1333	2 113 3593	1001
207	388·748 183 3537	2 103 1251	986
208	391·064 153 6991	2 092 9895	972
209	393·382 217 0341	2 082 9511	958
210	395·702 363 3202	2 073 0086	945
211	398·024 582 6149	2 063 1606	931
212	400·348 865 0702	2 053 4056	918
213	402·675 200 9312	2 043 7425	905
214	405·003 580 5346	2 034 1699	893
215	407·333 994 3080	2 024 6866	880
216	409·666 432 7679	2 015 2912	868
217	412·000 886 5190	2 005 9827	856
218	414·337 346 2529	1 996 7598	844
219	416·675 802 7465	1 987 6212	833
220	419·016 246 8613	1 978 5660	821

p	$\operatorname{Log}\Gamma(p)$	δ^2	δ^4
220	419·016 246 8613	1 978 5660	821
221	421·358 669 5421	1 969 5929	810
222	423·703 061 8158	1 960 7008	799
223	426·049 414 7903	1 951 8886	789
224	428·397 719 6533	1 943 1553	778
225	430·747 967 6717	1 934 4998	768
226	433·100 150 1898	1 925 9210	758
227	435·454 258 6289	1 917 4180	748
228	437·810 284 4861	1 908 9898	738
229	440·168 219 3331	1 900 6353	728
230	442·528 054 8154	1 892 3537	719
231	444·889 782 6515	1 884 1439	709
232	447·253 394 6314	1 876 0050	700
233	449·618 882 6162	1 867 9361	691
234	451·986 238 5373	1 859 9364	682
235	454·355 454 3947	1 852 0049	674
236	456·726 522 2570	1 844 1407	665
237	459·099 434 2599	1 836 3430	657
238	461·474 182 6059	1 828 6110	648
239	463·850 759 5630	1 820 9439	640
240	466·229 157 4639	1 813 3408	632
241	468·609 368 7056	1 805 8009	624
242	470·991 385 7482	1 798 3234	617
243	473·375 201 1142	1 790 9076	609
244	475·760 807 3878	1 783 5527	602
245	478·148 197 2141	1 776 2580	594
246	480·537 363 2985	1 769 0227	587
247	482·928 298 4056	1 761 8462	580
248	485·320 995 3589	1 754 7276	573
249	487·715 447 0397	1 747 6663	566
250	490·111 646 3868	1 740 6616	559
251	492·509 586 3955	1 733 7128	553
252	494·909 260 1169	1 726 8193	546
253	497·310 660 6577	1 719 9804	540
254	499·713 781 1789	1 713 1954	533
255	502·118 614 8955	1 706 4638	527
256	504·525 155 0760	1 699 7849	521
257	506·933 395 0413	1 693 1580	515
258	509·343 328 1646	1 686 5826	509
259	511·754 947 8706	1 680 0581	503
260	514·168 247 6346	1 673 5839	497
261	516·583 220 9826	1 667 1594	491
262	518·999 861 4900	1 660 7840	486
263	521·418 162 7813	1 654 4572	480
264	523·838 118 5298	1 648 1784	475
265	526·259 722 4566	1 641 9471	469
266	528·682 968 3306	1 635 7627	464
267	531·107 849 9672	1 629 6247	459
268	533·534 361 2286	1 623 5327	454
269	535·962 496 0226	1 617 4860	449
270	538·392 248 3026	1 611 4842	444

p	Log $\Gamma(p)$	δ^2	δ^4
270	538·392 248 3026	1 611 4842	444
271	540·823 612 0668	1 605 5267	439
272	543·256 581 3576	1 599 6132	434
273	545·691 150 2617	1 593 7430	429
274	548·127 312 9087	1 587 9158	425
275	550·565 063 4715	1 582 1310	420
276	553·004 396 1654	1 576 3882	415
277	555·445 305 2474	1 570 6870	411
278	557·887 785 0165	1 565 0269	406
279	560·331 829 8124	1 559 4074	402
280	562·777 434 0157	1 553 8281	398
281	565·224 592 0470	1 548 2886	394
282	567·673 298 3669	1 542 7884	389
283	570·123 547 4752	1 537 3272	385
284	572·575 333 9108	1 531 9045	381
285	575·028 652 2508	1 526 5200	377
286	577·483 497 1108	1 521 1731	373
287	579·939 863 1439	1 515 8636	369
288	582·397 745 0407	1 510 5910	366
289	584·857 137 5284	1 505 3550	362
290	587·318 035 3712	1 500 1551	358
291	589·780 433 3691	1 494 9911	354
292	592·244 326 3581	1 489 8625	351
293	594·709 709 2095	1 484 7689	347
294	597·176 576 8299	1 479 7101	344
295	599·644 924 1603	1 474 6856	340
296	602·114 746 1763	1 469 6951	337
297	604·586 037 8873	1 464 7383	333
298	607·058 794 3367	1 459 8148	330
299	609·533 010 6007	1 454 9242	327
300	612·008 681 7891	1 450 0664	323
301	614·485 803 0438	1 445 2409	320
302	616·964 369 5394	1 440 4474	317
303	619·444 376 4823	1 435 6855	314
304	621·925 819 1108	1 430 9551	311
305	624·408 692 6944	1 426 2557	308
306	626·892 992 5338	1 421 5871	305
307	629·378 713 9603	1 416 9490	302
308	631·865 852 3357	1 412 3410	299
309	634·354 403 0522	1 407 7629	296
310	636·844 361 5317	1 403 2144	293
311	639·335 723 2255	1 398 6952	290
312	641·828 483 6145	1 394 2050	287
313	644·322 638 2085	1 389 7435	285
314	646·818 182 5461	1 385 3105	282
315	649·315 112 1942	1 380 9057	279
316	651·813 422 7480	1 376 5288	277
317	654·313 109 8306	1 372 1796	274
318	656·814 169 0928	1 367 8578	271
319	659·316 596 2128	1 363 5631	269
320	661·820 386 8958	1 359 2953	266

p	Log $\Gamma(p)$	δ^2	δ^4
320	661·820 386 8958	1 359 2953	266
321	664·325 536 8742	1 355 0541	264
322	666·832 041 9066	1 350 8393	261
323	669·339 897 7783	1 346 6506	259
324	671·849 100 3006	1 342 4879	257
325	674·359 645 3108	1 338 3508	254
326	676·871 528 6718	1 334 2391	252
327	679·384 746 2718	1 330 1526	250
328	681·899 294 0245	1 326 0911	247
329	684·415 167 8682	1 322 0542	245
330	686·932 363 7662	1 318 0419	243
331	689·450 877 7060	1 314 0539	241
332	691·970 705 6998	1 310 0899	238
333	694·491 843 7835	1 306 1498	236
334	697·014 288 0170	1 302 2333	234
335	699·538 034 4838	1 298 3402	232
336	702·063 079 2909	1 294 4704	230
337	704·589 418 5683	1 290 6235	228
338	707·117 048 4691	1 286 7994	226
339	709·645 965 1694	1 282 9979	224
340	712·176 164 8676	1 279 2188	222
341	714·707 643 7847	1 275 4620	220
342	717·240 398 1636	1 271 7271	218
343	719·774 424 2697	1 268 0140	216
344	722·309 718 3897	1 264 3225	214
345	724·846 276 8323	1 260 6525	212
346	727·384 095 9274	1 257 0037	211
347	729·923 172 0262	1 253 3760	209
348	732·463 501 5010	1 249 7692	207
349	735·005 080 7449	1 246 1830	205
350	737·547 906 1719	1 242 6174	203
351	740·091 974 2162	1 239 0721	202
352	742·637 281 3327	1 235 5470	200
353	745·183 823 9962	1 232 0419	198
354	747·731 598 7016	1 228 5566	197
355	750·280 601 9636	1 225 0910	195
356	752·830 830 3166	1 221 6449	193
357	755·382 280 3146	1 218 2181	192
358	757·934 948 5307	1 214 8105	190
359	760·488 831 5574	1 211 4219	189
360	763·043 926 0060	1 208 0522	187
361	765·600 228 5067	1 204 7011	185
362	768·157 735 7086	1 201 3686	184
363	770·716 444 2792	1 198 0545	182
364	773·276 350 9042	1 194 7586	181
365	775·837 452 2878	1 191 4808	179
366	778·399 745 1523	1 188 2209	178
367	780·963 226 2377	1 184 9789	176
368	783·527 892 3019	1 181 7544	175
369	786·093 740 1206	1 178 5475	174
370	788·660 766 4868	1 175 3579	172

p	Log $\Gamma(p)$	δ^2	δ^4
370	788·660 766 4868	1 175 3579	172
371	791·228 968 2108	1 172 1855	171
372	793·798 342 1205	1 169 0303	169
373	796·368 885 0603	1 165 8919	168
374	798·940 593 8922	1 162 7704	167
375	801·513 465 4944	1 159 6655	165
376	804·087 496 7621	1 156 5772	164
377	806·662 684 6070	1 153 5053	163
378	809·239 025 9572	1 150 4496	161
379	811·816 517 7571	1 147 4101	160
380	814·395 156 9670	1 144 3866	159
381	816·974 940 5636	1 141 3791	158
382	819·555 865 5393	1 138 3872	156
383	822·137 928 9022	1 135 4111	155
384	824·721 127 6762	1 132 4504	154
385	827·305 458 9006	1 129 5051	153
386	829·890 919 6301	1 126 5752	152
387	832·477 506 9347	1 123 6603	150
388	835·065 217 8998	1 120 7606	149
389	837·654 049 6254	1 117 8757	148
390	840·243 999 2267	1 115 0057	147
391	842·835 063 8337	1 112 1504	146
392	845·427 240 5911	1 109 3096	145
393	848·020 526 6581	1 106 4834	144
394	850·614 919 2085	1 103 6715	143
395	853·210 415 4303	1 100 8738	141
396	855·807 012 5259	1 098 0903	140
397	858·404 707 7119	1 095 3208	139
398	861·003 498 2186	1 092 5653	138
399	863·603 381 2907	1 089 8236	137
400	866·204 354 1864	1 087 0956	136
401	868·806 414 1777	1 084 3813	135
402	871·409 558 5503	1 081 6805	134
403	874·013 784 6034	1 078 9931	133
404	876·619 089 6496	1 076 3190	132
405	879·225 471 0147	1 073 6581	131
406	881·832 926 0379	1 071 0104	130
407	884·441 452 0715	1 068 3756	129
408	887·051 046 4807	1 065 7539	128
409	889·661 706 6438	1 063 1449	127
410	892·273 429 9518	1 060 5487	126
411	894·886 213 8085	1 057 9652	126
412	897·500 055 6304	1 055 3942	125
413	900·114 952 8464	1 052 8356	124
414	902·730 902 8981	1 050 2895	123
415	905·347 903 2392	1 047 7556	122
416	907·965 951 3359	1 045 2339	121
417	910·585 044 6665	1 042 7243	120
418	913·205 180 7215	1 040 2268	119
419	915·826 357 0033	1 037 7412	119
420	918·448 571 0263	1 035 2674	118

p	Log $\Gamma(p)$	δ^2	δ^4
420	918·448 571 0263	1 035 2674	118
421	921·071 820 3167	1 032 8054	117
422	923·696 102 4125	1 030 3551	116
423	926·321 414 8635	1 027 9164	115
424	928·947 755 2308	1 025 4892	114
425	931·575 121 0874	1 023 0735	114
426	934·203 510 0175	1 020 6691	113
427	936·832 919 6166	1 018 2759	112
428	939·463 347 4916	1 015 8940	111
429	942·094 791 2606	1 013 5232	110
430	944·727 248 5528	1 011 1634	110
431	947·360 717 0084	1 008 8146	109
432	949·995 194 2785	1 006 4767	108
433	952·630 678 0254	1 004 1495	107
434	955·267 165 9217	1 001 8332	107
435	957·904 655 6512	999 5274	106
436	960·543 144 9082	997 2323	105
437	963·182 631 3974	994 9477	104
438	965·823 112 8344	992 6735	104
439	968·464 586 9449	990 4097	103
440	971·107 051 4652	988 1562	102
441	973·750 504 1416	985 9130	102
442	976·394 942 7311	983 6799	101
443	979·040 365 0005	981 4569	100
444	981·686 768 7267	979 2439	100
445	984·334 151 6968	977 0409	99
446	986·982 511 7078	974 8477	98
447	989·631 846 5665	972 6644	98
448	992·282 154 0896	970 4909	97
449	994·933 432 1036	968 3270	96
450	997·585 678 4446	966 1728	96
451	1000·238 890 9584	964 0281	95
452	1002·893 067 5003	961 8929	94
453	1005·548 205 9351	959 7672	94
454	1008·204 304 1371	957 6508	93
455	1010·861 359 9900	955 5438	93
456	1013·519 371 3866	953 4460	92
457	1016·178 336 2293	951 3574	91
458	1018·838 252 4293	949 2779	91
459	1021·499 117 9074	947 2075	90
460	1024·160 930 5929	945 1461	90
461	1026·823 688 4246	943 0937	89
462	1029·487 389 3500	941 0502	88
463	1032·152 031 3255	939 0155	88
464	1034·817 612 3165	936 9895	87
465	1037·484 130 2971	934 9723	87
466	1040·151 583 2500	932 9638	86
467	1042·819 969 1667	930 9639	86
468	1045·489 286 0472	928 9725	85
469	1048·159 531 9003	926 9896	84
470	1050·830 704 7430	925 0152	84

p	Log $\Gamma(p)$	δ^2	δ^4
470	1050·830 704 7430	925 0152	84
471	1053·502 802 6010	923 0492	83
472	1056·175 823 5081	921 0915	83
473	1058·849 765 5067	919 1421	82
474	1061·524 626 6475	917 2009	82
475	1064·200 404 9891	915 2680	81
476	1066·877 098 5988	913 3431	81
477	1069·554 705 5515	911 4263	80
478	1072·233 223 9305	909 5176	80
479	1074·912 651 8271	907 6168	79
480	1077·592 987 3405	905 7240	79
481	1080·274 228 5779	903 8390	78
482	1082·956 373 6543	901 9619	78
483	1085·639 420 6925	900 0925	77
484	1088·323 367 8233	898 2309	77
485	1091·008 213 1849	896 3770	76
486	1093·693 954 9235	894 5307	76
487	1096·380 591 1928	892 6920	75
488	1099·068 120 1540	890 8608	75
489	1101·756 539 9760	889 0371	75
490	1104·445 848 8351	887 2209	74
491	1107·136 044 9152	885 4121	74
492	1109·827 126 4073	883 6106	73
493	1112·519 091 5101	881 8165	73
494	1115·211 938 4293	880 0296	72
495	1117·905 665 3783	878 2500	72
496	1120·600 270 5772	876 4776	71
497	1123·295 752 2537	874 7122	71
498	1125·992 108 6424	872 9540	71
499	1128·689 337 9852	871 2029	70
500	1131·387 438 5308	869 4587	70
501	1134·086 408 5351	867 7215	69
502	1136·786 246 2610	865 9913	69
503	1139·486 949 9781	864 2679	68
504	1142·188 517 9632	862 5514	68
505	1144·890 948 4996	860 8417	68
506	1147·594 239 8778	859 1387	67
507	1150·298 390 3946	857 4425	67
508	1153·003 398 3539	855 7530	66
509	1155·709 262 0662	854 0701	66
510	1158·415 979 8486	852 3938	66
511	1161·123 550 0247	850 7240	65
512	1163·831 970 9248	849 0608	65
513	1166·541 240 8858	847 4041	65
514	1169·251 358 2509	845 7539	64
515	1171·962 321 3699	844 1100	64
516	1174·674 128 5989	842 4726	63
517	1177·386 778 3005	840 8415	63
518	1180·100 268 8436	839 2167	63
519	1182·814 598 6034	837 5981	62
520	1185·529 765 9612	835 9858	62

p	Log $\Gamma(p)$	δ^2	δ^4
520	1185·529 765 9612	835 9858	62
521	1188·245 769 3049	834 3797	62
522	1190·962 607 0282	832 7797	61
523	1193·680 277 5312	831 1859	61
524	1196·398 779 2200	829 5981	61
525	1199·118 110 5070	828 0164	60
526	1201·838 269 8104	826 4407	60
527	1204·559 255 5546	824 8711	60
528	1207·281 066 1698	823 3073	59
529	1210·003 700 0923	821 7495	59
530	1212·727 155 7644	820 1976	59
531	1215·451 431 6340	818 6515	58
532	1218·176 526 1550	817 1112	58
533	1220·902 437 7873	815 5767	58
534	1223·629 164 9964	814 0480	57
535	1226·356 706 2534	812 5250	57
536	1229·085 060 0354	811 0077	57
537	1231·814 224 8251	809 4960	56
538	1234·544 199 1108	807 9900	56
539	1237·274 981 3865	806 4895	56
540	1240·006 570 1517	804 9946	55
541	1242·738 963 9115	803 5053	55
542	1245·472 161 1766	802 0214	55
543	1248·206 160 4631	800 5431	54
544	1250·940 960 2927	799 0701	54
545	1253·676 559 1924	797 6026	54
546	1256·412 955 6947	796 1404	54
547	1259·150 148 3374	794 6836	53
548	1261·888 135 6637	793 2322	53
549	1264·626 916 2222	791 7860	53
550	1267·366 488 5667	790 3450	52
551	1270·106 851 2562	788 9094	52
552	1272·848 002 8550	787 4789	52
553	1275·589 941 9327	786 0536	52
554	1278·332 667 0640	784 6334	51
555	1281·076 176 8288	783 2184	51
556	1283·820 469 8119	781 8085	51
557	1286·565 544 6035	780 4036	50
558	1289·311 399 7987	779 0038	50
559	1292·058 033 9976	777 6089	50
560	1294·805 445 8055	776 2191	50
561	1297·553 633 8325	774 8342	49
562	1300·302 596 6937	773 4543	49
563	1303·052 333 0093	772 0793	49
564	1305·802 841 4042	770 7091	49
565	1308·554 120 5081	769 3438	48
566	1311·306 168 9560	767 9834	48
567	1314·058 985 3871	766 6277	48
568	1316·812 568 4460	765 2768	48
569	1319·566 916 7818	763 9307	47
570	1322·322 029 0481	762 5893	47

p	$\mathrm{Log}\,\Gamma(p)$	δ^2	δ^4
570	1322·322 029 0481	762 5893	47
571	1325·077 903 9038	761 2526	47
572	1327·834 540 0121	759 9205	47
573	1330·591 936 0409	758 5932	46
574	1333·350 090 6628	757 2704	46
575	1336·109 002 5552	755 9523	46
576	1338·868 670 3999	754 6387	46
577	1341·629 092 8833	753 3297	45
578	1344·390 268 6965	752 0253	45
579	1347·152 196 5349	750 7253	45
580	1349·914 875 0986	749 4298	45
581	1352·678 303 0922	748 1388	44
582	1355·442 479 2246	746 8523	44
583	1358·207 402 2092	745 5701	44
584	1360·973 070 7640	744 2924	44
585	1363·739 483 6111	743 0190	43
586	1366·506 639 4772	741 7499	43
587	1369·274 537 0932	740 4852	43
588	1372·043 175 1945	739 2248	43
589	1374·812 552 5205	737 9687	43
590	1377·582 667 8153	736 7169	42
591	1380·353 519 8270	735 4692	42
592	1383·125 107 3079	734 2258	42
593	1385·897 429 0146	732 9866	42
594	1388·670 483 7079	731 7516	42
595	1391 444 270 1529	730 5207	41
596	1394·218 787 1186	729 2940	41
597	1396·994 033 3784	728 0714	41
598	1399·770 007 7095	726 8529	41
599	1402·546 708 8935	725 6384	41
600	1405·324 135 7159	724 4280	40
601	1408·102 286 9663	723 2216	40
602	1410·881 161 4383	722 0193	40
603	1413·660 757 9295	720 8209	40
604	1416·441 075 2417	719 6265	40
605	1419·222 112 1803	718 4360	39
606	1422·003 867 5550	717 2495	39
607	1424·786 340 1791	716 0669	39
608	1427·569 528 8702	714 8882	39
609	1430·353 432 4495	713 7134	39
610	1433·138 049 7421	712 5424	38
611	1435·923 379 5771	711 3752	38
612	1438·709 420 7874	710 2119	38
613	1441·496 172 2095	709 0524	38
614	1444·283 632 6840	707 8966	38
615	1447·071 801 0552	706 7446	37
616	1449·860 676 1709	705 5964	37
617	1452·650 256 8831	704 4519	37
618	1455·440 542 0471	703 3111	37
619	1458·231 530 5222	702 1739	37
620	1461·023 221 1712	701 0405	37

p	$\mathrm{Log}\,\Gamma(p)$	δ^2	δ^4
620	1461·023 221 1712	701 0405	37
621	1463·815 612 8607	699 9107	36
622	1466·608 704 4609	698 7845	36
623	1469·402 494 8456	697 6620	36
624	1472·196 982 8923	696 5430	36
625	1474·992 167 4819	695 4277	36
626	1477·788 047 4993	694 3159	35
627	1480·584 621 8325	693 2076	35
628	1483·381 889 3733	692 1029	35
629	1486·179 849 0171	691 0017	35
630	1488·978 499 6625	689 9040	35
631	1491·777 840 2120	688 8098	35
632	1494·577 869 5712	687 7190	34
633	1497·378 586 6495	686 6317	34
634	1500·179 990 3595	685 5479	34
635	1502·982 079 6174	684 4674	34
636	1505·784 853 3427	683 3904	34
637	1508·588 310 4583	682 3167	34
638	1511·392 449 8907	681 2464	34
639	1514·197 270 5694	680 1794	33
640	1517·002 771 4275	679 1158	33
641	1519·808 951 4015	678 0555	33
642	1522·615 809 4311	676 9986	33
643	1525·423 344 4591	675 9449	33
644	1528·231 555 4320	674 8944	33
645	1531·040 441 2994	673 8473	32
646	1533·850 001 0140	672 8034	32
647	1536·660 233 5320	671 7627	32
648	1539·471 137 8127	670 7252	32
649	1542·282 712 8186	669 6909	32
650	1545·094 957 5154	668 6598	32
651	1547·907 870 8720	667 6319	32
652	1550·721 451 8606	666 6072	31
653	1553·535 699 4563	665 5855	31
654	1556·350 612 6376	664 5670	31
655	1559·166 190 3859	663 5517	31
656	1561·982 431 6859	662 5394	31
657	1564·799 335 5253	661 5302	31
658	1567·616 900 8948	660 5241	31
659	1570·435 126 7885	659 5210	30
660	1573·254 012 2031	658 5209	30
661	1576·073 556 1386	657 5239	30
662	1578·893 757 5981	656 5300	30
663	1581·714 615 5875	655 5390	30
664	1584·536 129 1159	654 5510	30
665	1587·358 297 1953	653 5659	30
666	1590·181 118 8406	652 5839	29
667	1593·004 593 0698	651 6047	29
668	1595·828 718 9037	650 6286	29
669	1598·653 495 3662	649 6553	29
670	1601·478 921 4839	648 6849	29

p	Log $\Gamma(p)$	δ^2	δ^4
670	1601·478 921 4839	648 6849	29
671	1604·304 996 2866	647 7175	29
672	1607·131 718 8068	646 7529	29
673	1609·959 088 0798	645 7912	29
674	1612·787 103 1441	644 8323	28
675	1615·615 763 0406	643 8763	28
676	1618·445 066 8134	642 9231	28
677	1621·275 013 5094	641 9727	28
678	1624·105 602 1781	641 0252	28
679	1626·936 831 8719	640 0804	28
680	1629·768 701 6462	639 1384	28
681	1632·601 210 5589	638 1992	28
682	1635·434 357 6708	637 2627	27
683	1638·268 142 0455	636 3290	27
684	1641·102 562 7492	635 3980	27
685	1643·937 618 8509	634 4698	27
686	1646·773 309 4224	633 5442	27
687	1649·609 633 5381	632 6214	27
688	1652·446 590 2752	631 7012	27
689	1655·284 178 7134	630 7837	27
690	1658·122 397 9353	629 8688	26
691	1660·961 247 0260	628 9566	26
692	1663·800 725 0734	628 0471	26
693	1666·640 831 1679	627 1402	26
694	1669·481 564 4025	626 2358	26
695	1672·322 923 8729	625 3341	26
696	1675·164 908 6775	624 4350	26
697	1678·007 517 9171	683 5385	26
698	1680·850 750 6952	622 6445	26
699	1683·694 606 1179	621 7531	25
700	1686·539 083 2936	620 8643	25
701	1689·384 181 3336	619 9780	25
702	1692·229 899 3516	619 0942	25
703	1695·076 236 4637	618 2129	25
704	1697·923 191 7887	617 3341	25
705	1700·770 764 4479	616 4578	25
706	1703·618 953 5649	615 5841	25
707	1706·467 758 2659	614 7127	25
708	1709·317 177 6797	613 8439	25
709	1712·167 210 9374	612 9775	24
710	1715·017 857 1726	612 1135	24
711	1717·869 115 5213	611 2520	24
712	1720·720 985 1220	610 3929	24
713	1723·573 465 1157	609 5362	24
714	1726·426 554 6455	608 6819	24
715	1729·280 252 8573	607 8300	24
716	1732·134 558 8991	606 9805	24
717	1734·989 471 9214	606 1334	24
718	1737·844 991 0771	605 2886	24
719	1740·701 115 5213	604 4461	23
720	1743·557 844 4117	603 6060	23

p	Log $\Gamma(p)$	δ^2	δ^4
720	1743·557 844 4117	603 6060	23
721	1746·415 176 9081	602 7683	23
722	1749·273 112 1728	601 9329	23
723	1752·131 649 3704	601 0997	23
724	1754·990 787 6677	600 2689	23
725	1757·850 526 2339	599 4404	23
726	1760·710 864 2405	598 6141	23
727	1763·571 800 8612	597 7902	23
728	1766·433 335 2720	596 9685	23
729	1769·295 466 6514	596 1490	22
730	1772·158 194 1797	595 3318	22
731	1775·021 517 0398	594 5168	22
732	1777·885 434 4167	593 7041	22
733	1780·749 945 4978	592 8936	22
734	1783·615 049 4724	592 0853	22
735	1786·480 745 5324	591 2792	22
736	1789·347 032 8714	590 4753	22
737	1792·213 910 6858	589 6735	22
738	1795·081 378 1736	588 8740	22
739	1797·949 434 5355	588 0766	22
740	1800·818 078 9739	587 2813	21
741	1803·687 310 6936	586 4882	21
742	1806·557 128 9016	585 6973	21
743	1809·427 532 8069	584 9085	21
744	1812·298 521 6206	584 1218	21
745	1815·170 094 5562	583 3372	21
746	1818·042 250 8289	582 5547	21
747	1820·914 989 6564	581 7743	21
748	1823·788 310 2582	580 9960	21
749	1826·662 211 8561	580 2198	21
750	1829·536 693 6738	579 4457	21
751	1832·411 754 9372	578 6736	21
752	1835·287 394 8742	577 9036	20
753	1838·163 612 7147	577 1356	20
754	1841·040 407 6909	576 3697	20
755	1843·917 779 0368	575 6058	20
756	1846·795 725 9884	574 8439	20
757	1849·674 247 7839	574 0840	20
758	1852·553 343 6634	573 3261	20
759	1855·433 012 8691	572 5703	20
760	1858·313 254 6450	571 8164	20
761	1861·194 068 2373	571 0645	20
762	1864·075 452 8940	570 3146	20
763	1866·957 407 8654	569 5666	20
764	1869·839 932 4033	568 8206	20
765	1872·723 025 7619	568 0766	19
766	1875·606 687 1971	567 3345	19
767	1878·490 915 9667	566 5943	19
768	1881·375 711 3306	565 8561	19
769	1884·261 072 5507	565 1198	19
770	1887·146 998 8905	564 3854	19

p	$\operatorname{Log}\Gamma(p)$	δ^2	δ^4
770	1887·146 998 8905	564 3854	19
771	1890·033 489 6156	563 6529	19
772	1892·920 543 9937	562 9223	19
773	1895·808 161 2940	562 1936	19
774	1898·696 340 7879	561 4668	19
775	1901·585 081 7486	560 7418	19
776	1904·474 383 4511	560 0188	19
777	1907·364 245 1724	559 2975	19
778	1910·254 666 1912	558 5782	18
779	1913·145 645 7882	557 8607	18
780	1916·037 183 2459	557 1450	18
781	1918·929 277 8485	556 4312	18
782	1921·821 928 8824	555 7192	18
783	1924·715 135 6355	555 0090	18
784	1927·608 897 3975	554 3006	18
785	1930·503 213 4602	553 5941	18
786	1933·398 083 1170	552 8893	18
787	1936·293 505 6630	552 1863	18
788	1939·189 480 3954	551 4851	18
789	1942·086 006 6129	550 7857	18
790	1944·983 083 6161	550 0881	18
791	1947·880 710 7074	549 3922	18
792	1950·778 887 1909	548 6981	18
793	1953·677 612 3724	548 0057	17
794	1956·576 885 5598	547 3151	17
795	1959·476 706 0622	546 6262	17
796	1962·377 073 1908	545 9391	17
797	1965·277 986 2586	545 2537	17
798	1968·179 444 5800	544 5700	17
799	1971·081 447 4713	543 8880	17
800	1973·983 994 2506	543 2077	17
801	1976·887 084 2376	542 5291	17
802	1979·790 716 7537	541 8522	17
803	1982·694 891 1220	541 1770	17
804	1985·599 606 6673	540 5035	17
805	1988·504 862 7160	539 8316	17
806	1991·410 658 5964	539 1614	17
807	1994·316 993 6382	538 4929	17
808	1997·223 867 1729	537 8261	16
809	2000·131 278 5337	537 1608	16
810	2003·039 227 0553	536 4973	16
811	2005·947 712 0742	535 8353	16
812	2008·856 732 9284	535 1750	16
813	2011·766 288 9576	534 5164	16
814	2014·676 379 5032	533 8593	16
815	2017·587 003 9081	533 2039	16
816	2020·498 161 5169	532 5500	16
817	2023·409 851 6756	531 8978	16
818	2026·322 073 7322	531 2471	16
819	2029·234 827 0358	530 5981	16
820	2032·148 110 9376	529 9506	16

p	$\operatorname{Log}\Gamma(p)$	δ^2	δ^4
820	2032·148 110 9376	529 9506	16
821	2035·061 924 7900	529 3047	16
822	2037·976 267 9471	528 6604	16
823	2040·891 139 7646	528 0177	16
824	2043·806 539 5998	527 3765	16
825	2046·722 466 8115	526 7369	15
826	2049·638 920 7601	526 0988	15
827	2052·555 900 8074	525 4622	15
828	2055·473 406 3170	524 8272	15
829	2058·391 436 6537	524 1938	15
830	2061·309 991 1843	523 5618	15
831	2064·229 069 2767	522 9314	15
832	2067·148 670 3005	522 3025	15
833	2070·068 793 6267	521 6751	15
834	2072·989 438 6282	521 0492	15
835	2075·910 604 6788	520 4248	15
836	2078·832 291 1543	519 8020	15
837	2081·754 497 4317	519 1806	15
838	2084·677 222 8897	518 5606	15
839	2087·600 466 9083	517 9422	15
840	2090·524 228 8692	517 3252	15
841	2093·448 508 1552	516 7097	15
842	2096·373 304 1510	516 0957	15
843	2099·298 616 2425	515 4831	15
844	2102·224 443 8171	514 8720	14
845	2105·150 786 2638	514 2623	14
846	2108·077 642 9727	513 6541	14
847	2111·005 013 3358	513 0473	14
848	2113·932 896 7461	512 4419	14
849	2116·861 292 5984	511 8380	14
850	2119·790 200 2886	511 2355	14
851	2122·719 619 2143	510 6344	14
852	2125·649 548 7744	510 0347	14
853	2128·579 988 3692	509 4364	14
854	2131·510 937 4003	508 8395	14
855	2134·442 395 2710	508 2440	14
856	2137·374 361 3857	507 6499	14
857	2140·306 835 1504	507 0572	14
858	2143·239 815 9723	506 4659	14
859	2146·173 303 2602	505 8760	14
860	2149·107 296 4240	505 2874	14
861	2152·041 794 8753	504 7002	14
862	2154·976 798 0267	504 1144	14
863	2157·912 305 2925	503 5299	14
864	2160·848 316 0883	502 9468	13
865	2163·784 829 8307	502 3650	13
866	2166·721 845 9382	501 7846	13
867	2169·659 363 8302	501 2055	13
868	2172·597 382 9277	500 6277	13
869	2175·535 902 6529	500 0513	13
870	2178·474 922 4293	499 4762	13

p	Log Γ(p)	δ^2	δ^4
870	2178·474 922 4293	499 4762	13
871	2181·414 441 6819	498 9024	13
872	2184·354 459 8370	498 3299	13
873	2187·294 976 3219	497 7588	13
874	2190·235 990 5656	497 1889	13
875	2193·177 501 9982	496 6204	13
876	2196·119 510 0512	496 0531	13
877	2199·062 014 1574	495 4872	13
878	2202·005 013 7508	494 9225	13
879	2204·948 508 2667	494 3592	13
880	2207·892 497 1418	493 7971	13
881	2210·836 979 8139	493 2363	13
882	2213·781 955 7223	492 6767	13
883	2216·727 424 3075	492 1184	13
884	2219·673 385 0110	491 5614	13
885	2222·619 837 2760	491 0057	13
886	2225·566 780 5467	490 4512	13
887	2228·514 214 2686	489 8979	12
888	2231·462 137 8885	489 3459	12
889	2234·410 550 8542	488 7952	12
890	2237·359 452 6152	488 2457	12
891	2240·308 842 6219	487 6974	12
892	2243·258 720 3259	487 1503	12
893	2246·209 085 1803	486 6045	12
894	2249·159 936 6392	486 0599	12
895	2252·111 274 1580	485 5165	12
896	2255·063 097 1933	484 9743	12
897	2258·015 405 2029	484 4334	12
898	2260·968 197 6460	483 8936	12
899	2263·921 473 9826	483 3551	12
900	2266·875 233 6744	482 8177	12
901	2269·829 476 1838	482 2815	12
902	2272·784 200 9748	481 7466	12
903	2275·739 407 5123	481 2128	12
904	2278·695 095 2626	480 6802	12
905	2281·651 263 6931	480 1487	12
906	2284·607 912 2723	479 6185	12
907	2287·565 040 4700	479 0894	12
908	2290·522 647 7571	478 5615	12
909	2293·480 733 6056	478 0347	12
910	2296·439 297 4888	477 5091	12
911	2299·398 338 8811	476 9847	12
912	2302·357 857 2581	476 4614	11
913	2305·317 852 0964	475 9392	11
914	2308·278 322 8740	475 4182	11
915	2311·239 269 0697	474 8983	11
916	2314·200 690 1638	474 3796	11
917	2317·162 585 6374	473 8620	11
918	2320·124 954 9731	473 3455	11
919	2323·087 797 6543	472 8302	11
920	2326·051 113 1657	472 3160	11

p	Log Γ(p)	δ^2	δ^4
920	2326·051 113 1657	472 3160	11
921	2329·014 900 9930	471 8029	11
922	2331·979 160 6232	471 2909	11
923	2334·943 891 5443	470 7800	11
924	2337·909 093 2453	470 2702	11
925	2340·874 765 2165	469 7615	11
926	2343·840 906 9493	469 2539	11
927	2346·807 517 9360	468 7475	11
928	2349·774 597 6701	468 2421	11
929	2352·742 145 6463	467 7378	11
930	2355·710 161 3603	467 2346	11
931	2358·678 644 3089	466 7324	11
932	2361·647 593 9898	466 2314	11
933	2364·617 009 9022	465 7314	11
934	2367·586 891 5459	465 2325	11
935	2370·557 238 4222	464 7346	11
936	2373·528 050 0331	464 2379	11
937	2376·499 325 8818	463 7421	11
938	2379·471 065 4727	463 2475	11
939	2382·443 268 3111	462 7539	11
940	2385·415 933 9033	462 2613	10
941	2388·389 061 7569	461 7698	10
942	2391·362 651 3803	461 2794	10
943	2394·336 702 2831	460 7899	10
944	2397·311 213 9759	460 3016	10
945	2400·286 185 9702	459 8142	10
946	2403·261 617 7787	459 3279	10
947	2406·237 508 9151	458 8426	10
948	2409·213 858 8941	458 3583	10
949	2412·190 667 2314	457 8751	10
950	2415·167 933 4439	457 3929	10
951	2418·145 657 0491	456 9116	10
952	2421·123 837 5661	456 4314	10
953	2424·102 474 5145	455 9523	10
954	2427·081 567 4151	455 4741	10
955	2430·061 115 7898	454 9969	10
956	2433·041 119 1614	454 5207	10
957	2436·021 577 0537	454 0455	10
958	2439·002 488 9914	453 5713	10
959	2441·983 854 5005	453 0981	10
960	2444·965 673 1077	452 6259	10
961	2447·947 944 3407	452 1546	10
962	2450·930 667 7284	451 6844	10
963	2453·913 842 8004	451 2151	10
964	2456·897 469 0876	450 7468	10
965	2459·881 546 1215	450 2794	10
966	2462·866 073 4348	449 8131	10
967	2465·851 050 5612	449 3477	10
968	2468·836 477 0353	448 8832	10
969	2471·822 352 3926	448 4197	10
970	2474·808 676 1697	447 9572	10

p	Log $\Gamma(p)$	δ^2	δ^4
970	2474·808 676 1697	447 9572	10
971	2477·795 447 9039	447 4956	10
972	2480·782 667 1338	447 0350	9
973	2483·770 333 3988	446 5753	9
974	2486·758 446 2390	446 1166	9
975	2489·747 005 1959	445 6588	9
976	2492·736 009 8116	445 2020	9
977	2495·725 459 6293	444 7461	9
978	2498·715 354 1930	444 2911	9
979	2501·705 693 0478	443 8370	9
980	2504·696 475 7396	443 3839	9
981	2507·687 701 8153	442 9317	9
982	2510·679 370 8227	442 4804	9
983	2513·671 482 3105	442 0300	9
984	2516·664 035 8283	441 5806	9
985	2519·657 030 9267	441 1321	9
986	2522·650 467 1572	440 6844	9
987	2525·644 344 0722	440 2377	9
988	2528·638 661 2248	439 7919	9
989	2531·633 418 1694	439 3470	9
990	2534·628 614 4610	438 9030	9
991	2537·624 249 6556	438 4599	9
992	2540·620 323 3101	438 0177	9
993	2543·616 834 9822	437 5763	9
994	2546·613 784 2307	437 1359	9
995	2549·611 170 6151	436 6965	9
996	2552·608 993 6959	436 2577	9
997	2555·607 253 0343	435 8199	9
998	2558·605 948 1926	435 3830	9
999	2561·605 078 7339	434 9469	9
1000	2564·604 644 2221	434 5118	9
1001	2567·604 644 2221	434 0775	9
1002	2570·605 078 2996	433 6441	9
1003	2573·605 946 0211	433 2115	9
1004	2576·607 246 9542	432 7798	9
1005	2579·608 980 6670	432 3489	9
1006	2582·611 146 7287	431 9190	9
1007	2585·613 744 7094	431 4898	9
1008	2588·616 774 1800	431 0616	8
1009	2591·620 234 7121	430 6341	8
1010	2594·624 125 8783	430 2075	8
1011	2597·628 447 2521	429 7818	8
1012	2600·633 198 4077	429 3569	8
1013	2603·638 378 9202	428 9329	8
1014	2606·643 988 3656	428 5096	8
1015	2609·650 026 3206	428 0873	8
1016	2612·656 492 3628	427 6657	8
1017	2615·663 386 0708	427 2450	8
1018	2618·670 707 0237	426 8251	8
1019	2621·678 454 8017	426 4060	8
1020	2624·686 628 9857	425 9878	8

p	Log $\Gamma(p)$	δ^2	δ^4
1020	2624·686 628 9857	425 9878	8
1021	2627·695 229 1575	425 5703	8
1022	2630·704 254 8996	425 1537	8
1023	2633·713 705 7954	424 7379	8
1024	2636·723 581 4291	424 3229	8
1025	2639·733 881 3857	423 9088	8
1026	2642·744 605 2511	423 4954	8
1027	2645·755 752 6119	423 0828	8
1028	2648·767 323 0555	422 6711	8
1029	2651·779 316 1701	422 2601	8
1030	2654·791 731 5449	421 8499	8
1031	2657·804 568 7696	421 4406	8
1032	2660·817 827 4349	421 0320	8
1033	2663·831 507 1322	420 6242	8
1034	2666·845 607 4537	420 2172	8
1035	2669·860 127 9925	419 8110	8
1036	2672·875 068 3422	419 4056	8
1037	2675·890 428 0977	419 0010	8
1038	2678·906 206 8540	418 5971	8
1039	2681·922 404 2076	418 1940	8
1040	2684·939 019 7551	417 7917	8
1041	2687·956 053 0944	417 3902	8
1042	2690·973 503 8239	416 9895	8
1043	2693·991 371 5429	416 5895	8
1044	2697·009 655 8513	416 1902	8
1045	2700·028 356 3500	415 7918	8
1046	2703·047 472 6404	415 3941	8
1047	2706·067 004 3250	414 9971	8
1048	2709·086 951 0066	414 6010	8
1049	2712·107 312 2893	414 2055	8
1050	2715·128 087 7775	413 8109	8
1051	2718·149 277 0765	413 4170	7
1052	2721·170 879 7926	413 0238	7
1053	2724·192 895 5324	412 6314	7
1054	2727·215 323 9036	412 2397	7
1055	2730·238 164 5145	411 8488	7
1056	2733·261 416 9741	411 4586	7
1057	2736·285 080 8923	411 0691	7
1058	2739·309 155 8796	410 6804	7
1059	2742·333 641 5473	410 2924	7
1060	2745·358 537 5074	409 9052	7
1061	2748·383 843 3727	409 5186	7
1062	2751·409 558 7566	409 1328	7
1063	2754·435 683 2733	408 7478	7
1064	2757·462 216 5378	408 3634	7
1065	2760·489 158 1658	407 9798	7
1066	2763·516 507 7736	407 5969	7
1067	2766·544 264 9783	407 2147	7
1068	2769·572 429 3977	406 8333	7
1069	2772·601 000 6504	406 4525	7
1070	2775·629 978 3556	406 0725	7

p	Log $\Gamma(p)$	δ^2	δ^4
1070	2775·629 978 3556	406 0725	7
1071	2778·659 362 1333	405 6931	7
1072	2781·689 151 6041	405 3145	7
1073	2784·719 346 3895	404 9366	7
1074	2787·749 946 1114	404 5594	7
1075	2790·780 950 3928	404 1829	7
1076	2793·812 358 8570	403 8071	7
1077	2796·844 171 1284	403 4320	7
1078	2799·876 386 8317	403 0576	7
1079	2802·909 005 5925	402 6838	7
1080	2805·942 027 0372	402 3108	7
1081	2808·975 450 7927	401 9385	7
1082	2812·009 276 4866	401 5668	7
1083	2815·043 503 7474	401 1959	7
1084	2818·078 132 2040	400 8256	7
1085	2821·113 161 4862	400 4560	7
1086	2824·148 591 2244	400 0871	7
1087	2827·184 421 0497	399 7188	7
1088	2830·220 650 5938	399 3513	7
1089	2833·257 279 4891	398 9844	7
1090	2836·294 307 3689	398 6182	7
1091	2839·331 733 8668	398 2526	7
1092	2842·369 558 6174	397 8878	7
1093	2845·407 781 2558	397 5236	7
1094	2848·446 401 4177	397 1600	7
1095	2851·485 418 7397	396 7972	7
1096	2854·524 832 8589	396 4350	7
1097	2857·564 643 4131	396 0734	7
1098	2860·604 850 0406	395 7125	7
1099	2863·645 452 3807	395 3523	7
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1103	2875·811 811 6718	393 9179	6
1104	·2878·854 387 1843	393 5610	6
1105	2881·897 356 2576	393 2046	6
1106	2884·940 718 5357	392 8489	6
1107	2887·984 473 6626	392 4939	6
1108	2891·028 621 2835	392 1395	6
1109	2894·073 161 0439	391 7858	6
1110	2897·118 092 5901	391 4326	6
1111	2900·163 415 5688	391 0802	6
1112	2903·209 129 6278	390 7283	6
1113	2906·255 234 4150	390 3771	6
1114	2909·301 729 5794	390 0265	6
1115	2912·348 614 7702	389 6765	6
1116	2915·395 889 6376	389 3272	6
1117	2918·443 553 8322	388 9785	6
1118	2921·491 607 0053	388 6304	6
1119	2924·540 048 8089	388 2830	6
1120	2927·588 878 8954	387 9361	6

p	Log $\Gamma(p)$	δ^2	δ^4
1120	2927·588 878 8954	387 9361	6
1121	2930·638 096 9181	387 5899	6
1122	2933·687 702 5306	387 2443	6
1123	2936·737 695 3876	386 8993	6
1124	2939·788 075 1438	386 5550	6
1125	2942·838 841 4551	386 2112	6
1126	2945·889 993 9775	385 8681	6
1127	2948·941 532 3680	385 5255	6
1128	2951·993 456 2841	385 1836	6
1129	2955·045 765 3837	384 8423	6
1130	2958·098 459 3256	384 5016	6
1131	2961·151 537 7691	384 1614	6
1132	2964·205 000 3741	383 8219	6
1133	2967·258 846 8009	383 4830	6
1134	2970·313 076 7108	383 1447	6
1135	2973·367 689 7653	382 8070	6
1136	2976·422 685 6269	382 4698	6
1137	2979·478 063 9582	382 1333	6
1138	2982·533 824 4229	381 7974	6
1139	2985·589 966 6850	381 4620	6
1140	2988·646 490 4091	381 1273	6
1141	2991·703 395 2604	380 7931	6
1142	2994·760 680 9048	380 4595	6
1143	2997·818 347 0087	380 1265	6
1144	3000·876 393 2391	379 7941	6
1145	3003·934 819 2636	379 4622	6
1146	3006·993 624 7502	379 1310	6
1147	3010·052 809 3679	378 8003	6
1148	3013·112 372 7858	378 4702	6
1149	3016·172 314 6738	378 1406	6
1150	3019·232 634 7025	377 8117	6
1151	3022·293 332 5429	377 4833	6
1152	3025·354 407 8665	377 1555	6
1153	3028·415 860 3456	376 8282	6
1154	3031·477 689 6529	376 5015	6
1155	3034·539 895 4617	376 1754	6
1156	3037·602 477 4459	375 8499	6
1157	3040·665 435 2800	375 5249	6
1158	3043·728 768 6390	375 2004	6
1159	3046·792 477 1984	374 8766	6
1160	3049·856 560 6343	374 5533	6
1161	3052·921 018 6236	374 2305	6
1162	3055·985 850 8433	373 9083	6
1163	3059·051 056 9714	373 5867	6
1164	3062·116 636 6861	373 2656	6
1165	3065·182 589 6664	372 9450	6
1166	3068·248 915 5918	372 6251	5
1167	3071·315 614 1422	372 3056	5
1168	3074·382 684 9982	371 9867	5
1169	3077·450 127 8410	371 6684	5
1170	3080·517 942 3522	371 3506	5

Complete Γ-Functions.

p	Log $\Gamma(p)$	δ^2	δ^4
1170	3080·517 942 3522	371 3506	5
1171	3083·586 128 2139	371 0333	5
1172	3086·654 685 1090	370 7166	5
1173	3089·723 612 7207	370 4004	5
1174	3092·792 910 7328	370 0848	5
1175	3095·862 578 8297	369 7697	5
1176	3098·932 616 6963	369 4551	5
1177	3102·003 024 0180	369 1411	5
1178	3105·073 800 4809	368 8276	5
1179	3108·144 945 7713	368 5146	5
1180	3111·216 459 5764	368 2022	5
1181	3114·288 341 5837	367 8903	5
1182	3117·360 591 4813	367 5789	5
1183	3120·433 208 9579	367 2681	5
1184	3123·506 193 7025	366 9578	5
1185	3126·579 545 4049	366 6480	5

p	Log $\Gamma(p)$	δ^2	δ^4
1185	3126·579 545 4049	366 6480	5
1186	3129·653 263 7553	366 3387	5
1187	3132·727 348 4443	366 0299	5
1188	3135·801 799 1632	365 7217	5
1189	3138·876 615 6039	365 4140	5
1190	3141·951 797 4585	365 1068	5
1191	3145·027 344 4199	364 8001	5
1192	3148·103 256 1814	364 4939	5
1193	3151·179 532 4368	364 1883	5
1194	3154·256 172 8805	363 8831	5
1195	3157·333 177 2072	363 5785	5
1196	3160·410 545 1125	363 2744	5
1197	3163·488 276 2922	362 9708	5
1198	3166·566 370 4426	362 6676	5
1199	3169·644 827 2606	362 3650	5
1200	3172·723 646 4437	362 0629	5

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PREFATORY NOTE

THE following table, due to Dr John Brownlee's initiative, rounds off the work of this series of *Tracts* on the Γ-function. It will enable the computer to obtain somewhat more easily (but to 7 instead of 10 figures) the value of the Γ-function for values of the argument lying between 1 and 50·9.

K. P.

The following Table of log Γ (x) gives the values to seven figures for values of x from unity to fifty-one. The compact form of this Table is rendered possible by the introduction of formulae for interpolating by means of central differences. Though the first differences are of six figures, from the beginning of the table to the end, the fourth difference is zero. The method of construction has been as follows. The values of log Γ (x) were taken to ten places of decimals from the facsimile of *Legendre's* tables published lately by Professor Pearson in the Tracts for Computers Series. The subsequent values were obtained by adding on a Burroughs' Adding Machine the logarithms of the appropriate numbers as given by Vega, in the *Thesaurus Logarithmorum Completus*, and the consecutive additions were carried on to the limit of the table. The final figures were then checked by the ordinary *Euler-Maclaurin* formula for log Γ (x). It was found in every case, in spite of the fact that fifty odd consecutive additions had been made, that at least eight figures of the results were correct, while in a number of cases the ninth was also accurate.

The whole of the work was carried out under the supervision of Mr W. T. Russell, chief statistical clerk to the National Institute for Medical Research, who himself checked all the differences. The whole credit for the table is thus his.

JOHN BROWNLEE.

National Institute for Medical Research.
Feb. 1*st*, 1923.

TABLE

OF

Log $\Gamma(x)$ from $x = 1$ to 50·9 by Intervals of ·01

	0		1		2		3		4	
1·0	0·0000000	714	9·9975287	704	9·9951279	694	9·9927964	684	9·9905334	675
1·1	9·9783407	622	9·9765313	615	9·9747834	607	9·9730962	599	9·9714689	592
1·2	9·9629225	550	9·9616946	544	9·9605212	538	9·9594015	532	9·9583350	526
1·3	9·9530203	493	9·9523100	487	9·9516485	482	9·9510353	477	9·9504698	473
1·4	9·9480528	445	9·9478084	441	9·9476081	437	9·9474515	433	9·9473382	429
1·5	9·9475449	406	9·9477237	402	9·9479426	399	9·9482015	394	9·9484998	392
1·6	9·9511020	373	9·9516680	370	9·9522710	367	9·9529107	364	9·9535867	361
1·7	9·9583912	344	9·9593141	342	9·9602712	339	9·9612622	337	9·9622869	334
1·8	9·9691287	320	9·9703823	318	9·9716678	316	9·9729848	313	9·9743331	311
1·9	9·9830693	299	9·9846311	297	9·9862226	295	9·9878436	293	9·9894938	291
2·0	·0000000	280	·0018501	278	·0037280	277	·0056336	274	·0075667	273
2·1	·0197334	264	·0218543	263	·0240014	260	·0261746	259	·0283737	257
2·2	·0421038	249	·0444800	247	·0468810	246	·0493066	245	·0517567	243
2·3	·0669636	236	·0695813	234	·0722225	233	·0748869	232	·0775746	231
2·4	·0941808	224	·0970275	223	·0998964	222	·1027875	220	·1057007	219
2·5	·1236362	213	·1267006	212	·1297862	211	·1328929	210	·1360206	209
2·6	·1552220	203	·1584939	202	·1617860	201	·1650983	200	·1684306	200
2·7	·1888402	194	·1923102	193	·1957997	193	·1993083	192	·2028362	191
2·8	·2244012	186	·2280609	185	·2317392	184	·2354359	184	·2391510	183
2·9	·2618229	178	·2656645	178	·2695238	177	·2734009	176	·2772956	176
3·0	·3010300	172	·3050462	171	·3090794	170	·3131297	170	·3171969	169
3·1	·3419527	165	·3461368	164	·3503373	164	·3545542	163	·3587875	163
3·2	·3845264	159	·3888723	159	·3932340	158	·3976115	157	·4020047	157
3·3	·4286915	154	·4331933	153	·4377105	152	·4422428	152	·4467904	151
3·4	·4743920	148	·4790445	148	·4837118	147	·4883938	147	·4930905	146
3·5	·5215762	143	·5263743	143	·5311867	143	·5360134	142	·5408543	142
3·6	·5701953	139	·5751344	138	·5800873	138	·5850540	138	·5900345	137
3·7	·6202039	135	·6252795	134	·6303686	134	·6354710	133	·6405867	133
3·8	·6715592	131	·6767672	130	·6819883	130	·6872223	129	·6924693	129
3·9	·7242209	127	·7295575	126	·7349067	126	·7402685	126	·7456429	125
4·0	·7781513	123	·7836127	123	·7890864	123	·7945723	122	·8000705	122
4·1	·8333143	120	·8388971	120	·8444919	119	·8500986	119	·8557171	119
4·2	·8896764	117	·8953773	116	·9010898	116	·9068140	116	·9125497	115
4·3	·9472054	114	·9530213	113	·9588485	113	·9646871	113	·9705369	112
4·4	1·0058710	111	1·0117989	110	1·0177379	110	1·0236879	110	1·0296489	110
4·5	1·0656443	108	1·0716814	108	1·0777294	107	1·0837881	107	1·0898575	107
4·6	1·1264978	105	1·1326416	105	1·1387959	105	1·1449607	105	1·1511359	104
4·7	1·1884057	103	1·1946535	103	1·2009115	102	1·2071798	102	1·2134583	102
4·8	1·2513428	101	1·2576922	100	1·2640516	100	1·2704211	100	1·2768005	100
4·9	1·3152856	98	1·3217343	98	1·3281928	98	1·3346611	98	1·3411391	97
5·0	1·3802112	96	1·3867570	96	1·3933124	96	1·3998774	95	1·4064519	95
5·1	1·4460982	94	1·4527390	94	1·4593891	94	1·4660486	93	1·4727175	93
5·2	1·5129257	92	1·5196594	92	1·5264023	92	1·5331544	91	1·5399156	91
5·3	1·5806739	90	1·5874986	90	1·5943323	90	1·6011750	90	1·6080266	89
5·4	1·6493236	88	1·6562375	88	1·6631602	88	1·6700916	88	1·6770319	88
5·5	1·7188568	87	1·7258580	86	1·7328678	86	1·7398863	86	1·7469134	86
5·6	1·7892557	85	1·7963425	85	1·8034379	85	1·8105416	84	1·8176539	84
5·7	1·8605035	83	1·8676744	83	1·8748535	83	1·8820410	83	1·8892367	83
5·8	1·9325840	82	1·9398373	82	1·9470987	81	1·9543682	81	1·9616459	81
5·9	2·0054816	80	2·0128157	80	2·0201579	80	2·0275080	80	2·0348661	80
6·0	2·0791812	79	2·0865948	79	2·0940161	78	2·1014453	78	2·1088824	78

5		6		7		8		9		
9·9883379	666	9·9862089	656	9·9841455	648	9·9821469	639	9·9802123	631	1·0
9·9699007	584	9·9683910	577	9·9669390	570	9·9655440	564	9·9642054	557	1·1
9·9573211	520	9·9563592	514	9·9554487	509	9·9545891	503	9·9537798	498	1·2
9·9499515	468	9·9494800	463	9·9490549	459	9·9486756	454	9·9483417	450	1·3
9·9472677	425	9·9472397	421	9·9472539	417	9·9473097	413	9·9474068	410	1·4
9·9488374	389	9·9492139	386	9·9496289	382	9·9500822	379	9·9505733	376	1·5
9·9542989	358	9·9550468	355	9·9558303	353	9·9566491	350	9·9575028	347	1·6
9·9633451	332	9·9644364	329	9·9655606	327	9·9667176	325	9·9679070	322	1·7
9·9757126	309	9·9771230	307	9·9785640	305	9·9800356	303	9·9815374	301	1·8
9·9911732	289	9·9928815	287	9·9946185	285	9·9963840	284	9·9981779	282	1·9
·0095272	272	·0115147	270	·0135293	268	·0155707	267	·0176388	265	2·0
·0305985	256	·0328490	255	·0351248	253	·0374260	252	·0397524	250	2·1
·0542311	242	·0567297	242	·0592524	239	·0617990	238	·0643695	237	2·2
·0802853	230	·0830190	228	·0857755	227	·0885547	226	·0913565	225	2·3
·1086357	218	·1115926	217	·1145712	216	·1175714	215	·1205931	214	2·4
·1391691	208	·1423385	207	·1455286	206	·1487392	205	·1519704	204	2·5
·1717828	199	·1751549	198	·1785468	197	·1819584	196	·1853895	195	2·6
·2063831	190	·2099490	189	·2135339	188	·2171376	188	·2207600	187	2·7
·2428843	182	·2466359	181	·2504056	181	·2541934	180	·2579992	179	2·8
·2812078	175	·2851375	174	·2890847	174	·2930492	173	·2970310	172	2·9
·3212810	168	·3253820	168	·3294996	167	·3336340	166	·3377851	166	3·0
·3630370	162	·3673027	161	·3715846	161	·3758825	160	·3801965	160	3·1
·4064136	156	·4108382	156	·4152783	155	·4197339	155	·4242050	154	3·2
·4513531	151	·4559310	150	·4605238	150	·4651316	149	·4697544	149	3·3
·4978018	146	·5025277	145	·5072681	145	·5120231	144	·5167924	144	3·4
·5457093	141	·5505785	141	·5554617	140	·5603589	140	·5652702	139	3·5
·5950287	137	·6000366	136	·6050581	136	·6100931	135	·6151418	135	3·6
·6457158	133	·6508581	132	·6560137	132	·6611824	131	·6663642	131	3·7
·6977292	129	·7030019	128	·7082875	128	·7135859	128	·7188971	127	3·8
·7510298	125	·7564292	125	·7618411	124	·7672655	124	·7727022	124	3·9
·8055809	122	·8111034	121	·8166380	121	·8221848	121	·8277435	120	4·0
·8613476	118	·8669898	118	·8726438	118	·8783096	117	·8839872	117	4·1
·9182970	115	·9240558	115	·9298260	115	·9356077	114	·9414009	114	4·2
·9763980	112	·9822702	112	·9881537	112	·9940483	111	·9999541	111	4·3
1·0356209	109	1·0416038	109	1·0475976	109	1·0536023	109	1·0596179	108	4·4
1·0959377	107	1·1020285	106	1·1081299	106	1·1142420	106	1·1203646	106	4·5
1·1573216	104	1·1635176	104	1·1697241	104	1·1759410	103	1·1821682	103	4·6
1·2197471	102	1·2260460	101	1·2323550	101	1·2386742	101	1·2450035	101	4·7
1·2831899	99	1·2895893	99	1·2959985	99	1·3024177	99	1·3088467	99	4·8
1·3476269	97	1·3541244	97	1·3606316	97	1·3671485	97	1·3736751	96	4·9
1·4130359	95	1·4196294	95	1·4262324	95	1·4328449	94	1·4394668	94	5·0
1·4793956	93	1·4860831	93	1·4927799	93	1·4994859	92	1·5062012	92	5·1
1·5466859	91	1·5534654	91	1·5602539	91	1·5670515	91	1·5738582	90	5·2
1·6148872	89	1·6217567	89	1·6286351	89	1·6355224	89	1·6424186	89	5·3
1·6839809	87	1·6909387	87	1·6979051	87	1·7048803	87	1·7118642	87	5·4
1·7539491	86	1·7609933	86	1·7680461	85	1·7751075	85	1·7821773	85	5·5
1·8247745	84	1·8319036	84	1·8390410	84	1·8461868	84	1·8533410	83	5·6
1·8964407	82	1·9036529	82	1·9108734	82	1·9181021	82	1·9253390	82	5·7
1·9689317	81	1·9762255	81	1·9835275	81	1·9908375	81	1·9981555	80	5·8
2·0422321	79	2·0496061	79	2·0569880	79	2·0643779	79	2·0717756	79	5·9
2·1163273	78	2·1237799	78	2·1312404	78	2·1387086	78	2·1461846	78	6·0

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6·0	2·0791812	79	2·0865948	79	2·0940161	78	2·1014453	78	2·1088824	78
6·1	2·1536684	77	2·1611599	77	2·1686591	77	2·1761660	77	2·1836806	77
6·2	2·2289290	76	2·2364971	76	2·2440728	76	2·2516560	76	2·2592469	75
6·3	2·3049497	75	2·3125931	75	2·3202439	74	2·3279022	74	2·3355679	74
6·4	2·3817174	73	2·3894348	73	2·3971595	73	2·4048915	73	2·4126308	73
6·5	2·4592195	72	2·4670096	72	2·4748069	72	2·4826114	72	2·4904232	72
6·6	2·5374437	71	2·5453054	71	2·5531742	71	2·5610500	71	2·5689330	71
6·7	2·6163784	70	2·6243105	70	2·6322495	70	2·6401956	70	2·6481486	69
6·8	2·6960120	69	2·7040134	69	2·7120217	69	2·7200368	68	2·7280587	68
6·9	2·7763336	68	2·7844032	68	2·7924796	68	2·8005627	67	2·8086525	67
7·0	2·8573325	67	2·8654692	67	2·8736126	66	2·8817627	66	2·8899193	66
7·1	2·9389982	66	2·9472011	66	2·9554105	65	2·9636265	65	2·9718490	65
7·2	3·0213207	65	3·0295887	65	3·0378632	65	3·0461441	64	3·0544314	64
7·3	3·1042903	64	3·1126225	64	3·1209610	64	3·1293059	63	3·1376571	63
7·4	3·1878974	63	3·1962928	63	3·2046945	63	3·2131024	63	3·2215167	62
7·5	3·2721328	62	3·2805906	62	3·2890545	62	3·2975246	62	3·3060009	62
7·6	3·3569876	61	3·3655069	61	3·3740322	61	3·3825636	61	3·3911011	61
7·7	3·4424532	60	3·4510330	60	3·4596188	60	3·4682106	60	3·4768085	60
7·8	3·5285209	59	3·5371605	59	3·5458060	59	3·5544575	59	3·5631148	59
7·9	3·6151827	59	3·6238813	59	3·6325857	58	3·6412959	58	3·6500120	58
8·0	3·7024305	58	3·7111872	58	3·7199497	58	3·7287180	58	3·7374920	58
8·1	3·7902566	57	3·7990707	57	3·8078905	57	3·8167160	57	3·8255472	57
8·2	3·8786532	56	3·8875240	56	3·8964004	56	3·9052824	56	3·9141700	56
8·3	3·9676131	56	3·9765398	56	3·9854721	55	3·9944099	55	4·0033532	55
8·4	4·0571291	55	4·0661110	55	4·0750984	55	4·0840913	55	4·0930896	55
8·5	4·1471941	54	4·1562305	54	4·1652724	54	4·1743196	54	4·1833723	54
8·6	4·2378012	54	4·2468915	53	4·2559871	53	4·2650881	53	4·2741944	53
8·7	4·3289439	53	4·3380874	53	4·3472361	53	4·3563901	53	4·3655494	53
8·8	4·4206155	52	4·4298116	52	4·4390128	52	4·4482192	52	4·4574309	52
8·9	4·5128098	52	4·5220578	52	4·5313108	52	4·5405691	51	4·5498325	51
9·0	4·6055205	51	4·6148198	51	4·6241241	51	4·6334335	51	4·6427480	51
9·1	4·6987416	50	4·7080915	50	4·7174465	50	4·7268065	50	4·7361716	50
9·2	4·7924671	50	4·8018672	50	4·8112722	50	4·8206822	50	4·8300972	50
9·3	4·8866912	49	4·8961409	49	4·9055954	49	4·9150549	49	4·9245192	49
9·4	4·9814084	49	4·9909070	49	5·0004105	49	5·0099188	49	5·0194320	49
9·5	5·0766130	48	5·0861601	48	5·0957119	48	5·1052686	48	5·1148301	48
9·6	5·1722997	48	5·1818947	48	5·1914944	48	5·2010989	48	5·2107082	47
9·7	5·2684632	47	5·2781055	47	5·2877526	47	5·2974044	47	5·3070609	47
9·8	5·3650982	47	5·3747875	47	5·3844814	47	5·3941799	47	5·4038832	46
9·9	5·4621998	46	5·4719355	46	5·4816757	46	5·4914205	46	5·5011700	46
10·0	5·5597630	46	5·5695446	46	5·5793306	46	5·5891213	46	5·5989165	45
10·1	5·6577830	45	5·6676099	45	5·6774414	45	5·6872773	45	5·6971178	45
10·2	5·7562549	45	5·7661268	45	5·7760031	45	5·7858839	45	5·7957692	45
10·3	5·8551742	44	5·8650905	44	5·8750113	44	5·8849365	44	5·8948661	44
10·4	5·9545362	44	5·9644966	44	5·9744614	44	5·9844305	44	5·9944040	44
10·5	6·0543366	43	6·0643406	43	6·0743489	43	6·0843615	43	6·0943785	43
10·6	6·1545709	43	6·1646181	43	6·1746695	43	6·1847252	43	6·1947852	43
10·7	6·2552349	43	6·2653247	43	6·2754189	42	6·2855172	42	6·2956198	42
10·8	6·3563243	42	6·3664565	42	6·3765929	42	6·3867335	42	6·3968783	42
10·9	6·4578350	42	6·4680091	42	6·4781874	42	6·4883698	42	6·4985564	42
11·0	6·5597630	41	6·5699786	41	6·5801984	41	6·5904222	41	6·6006502	41

5		6		7		8		9		
2·1163273	78	2·1237799	78	2·1312404	78	2·1387086	78	2·1461846	78	6·0
2·1912029	77	2·1987328	77	2·2062704	76	2·2138157	76	2·2213686	76	6·1
2·2668452	75	2·2744511	75	2·2820645	75	2·2896854	75	2·2973138	75	6·2
2·3432410	74	2·3509215	74	2·3586094	74	2·3663047	74	2·3740074	74	6·3
2·4203774	73	2·4281313	73	2·4358925	73	2·4436609	72	2·4514366	72	6·4
2·4982421	72	2·5060681	72	2·5139013	71	2·5217417	71	2·5295891	71	6·5
2·5768230	70	2·5847200	70	2·5926241	70	2·6005352	70	2·6084533	70	6·6
2·6561085	69	2·6640754	69	2·6720492	69	2·6800299	69	2·6880175	69	6·7
2·7360875	68	2·7441231	68	2·7521656	68	2·7602148	68	2·7682708	68	6·8
2·8167491	67	2·8248524	67	2·8329624	67	2·8410790	67	2·8492024	67	6·9
2·8980826	66	2·9062526	66	2·9144291	66	2·9226122	66	2·9308019	66	7·0
2·9800780	65	2·9883135	65	2·9965556	65	3·0048042	65	3·0130592	65	7·1
3·0627252	64	3·0710254	64	3·0793320	64	3·0876451	64	3·0959645	64	7·2
3·1460147	63	3·1543786	63	3·1627489	63	3·1711254	63	3·1795082	63	7·3
3·2299371	62	3·2383638	62	3·2467967	62	3·2552359	62	3·2636813	62	7·4
3·3144834	62	3·3229719	61	3·3314667	61	3·3399675	61	3·3484745	61	7·5
3·3996446	61	3·4081942	61	3·4167499	60	3·4253116	60	3·4338794	60	7·6
3·4854123	60	3·4940221	60	3·5026379	60	3·5112596	60	3·5198873	59	7·7
3·5717781	59	3·5804473	59	3·5891223	59	3·5978032	59	3·6064901	59	7·8
3·6587339	58	3·6674616	58	3·6761951	58	3·6849345	58	3·6936796	58	7·9
3·7462718	57	3·7550573	57	3·7638485	57	3·7726455	57	3·7814482	57	8·0
3·8343840	57	3·8432266	57	3·8520748	57	3·8609286	56	3·8697881	56	8·1
3·9230632	56	3·9319621	56	3·9408665	56	3·9497764	56	3·9586920	56	8·2
4·0123021	55	4·0212564	55	4·0302163	55	4·0391818	55	4·0481527	55	8·3
4·1020934	55	4·1111026	54	4·1201173	54	4·1291375	54	4·1381631	54	8·4
4·1924303	54	4·2014937	54	4·2105626	54	4·2196368	54	4·2287163	54	8·5
4·2833060	53	4·2924230	53	4·3015453	53	4·3106728	53	4·3198057	53	8·6
4·3747140	53	4·3838838	53	4·3930589	52	4·4022392	52	4·4114248	52	8·7
4·4666477	52	4·4758698	52	4·4850970	52	4·4943295	52	4·5035671	52	8·8
4·5591010	51	4·5683747	51	4·5776535	51	4·5869374	51	4·5962263	51	8·9
4·6520676	51	4·6613923	51	4·6707220	51	4·6800568	51	4·6893967	51	9·0
4·7455416	50	4·7549167	50	4·7642968	50	4·7736819	50	4·7830720	50	9·1
4·8395172	50	4·8489421	50	4·8583720	49	4·8678068	49	4·8772465	49	9·2
4·9339885	49	4·9434627	49	4·9529418	49	4·9624258	49	4·9719146	49	9·3
5·0289501	48	5·0384730	48	5·0480008	48	5·0575333	48	5·0670708	48	9·4
5·1243964	48	5·1339675	48	5·1435434	48	5·1531240	48	5·1627095	48	9·5
5·2203221	47	5·2299409	47	5·2395644	47	5·2491926	47	5·2588255	47	9·6
5·3167221	47	5·3263879	47	5·3360585	47	5·3457337	47	5·3554136	47	9·7
5·4135910	46	5·4233035	46	5·4330207	46	5·4427424	46	5·4524688	46	9·8
5·5109240	46	5·5206827	46	5·5304459	46	5·5402137	46	5·5499861	46	9·9
5·6087162	45	5·6185205	45	5·6283293	45	5·6381427	45	5·6479606	45	10·0
5·7069627	45	5·7168122	45	5·7266661	45	5·7365246	45	5·7463875	45	10·1
5·8056589	45	5·8155531	44	5·8254517	44	5·8353548	44	5·8452623	44	10·2
5·9048001	44	5·9147386	44	5·9246814	44	5·9346286	44	5·9445802	44	10·3
6·0043819	44	6·0143641	44	6·0243507	44	6·0343417	43	6·0443370	43	10·4
6·1043998	43	6·1144254	43	6·1244553	43	6·1344895	43	6·1445281	43	10·5
6·2048495	43	6·2149180	43	6·2249908	43	6·2350679	43	6·2451493	43	10·6
6·3057267	42	6·3158377	42	6·3259530	42	6·3360726	42	6·3461963	42	10·7
6·4070272	42	6·4171804	42	6·4273378	42	6·4374994	42	6·4476651	42	10·8
6·5087471	42	6·5189420	41	6·5291411	41	6·5393442	41	6·5495516	41	10·9
6·6108823	41	6·6211185	41	6·6313588	41	6·6416032	41	6·6518517	41	11·0

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11·0	6·5597630	41	6·5699786	41	6·5801984	41	6·5904222	41	6·6006502	41
11·1	6·6621043	41	6·6723611	41	6·6826219	41	6·6928868	41	6·7031557	41
11·2	6·7648551	41	6·7751525	41	6·7854540	40	6·7957596	40	6·8060692	40
11·3	6·8680114	40	6·8783492	40	6·8886910	40	6·8990368	40	6·9093867	40
11·4	6·9715696	40	6·9819474	40	6·9923291	40	7·0027148	40	7·0131045	40
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11·7	7·2846187	39	7·2951142	39	7·3056136	39	7·3161169	39	7·3266241	39
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11·9	7·4952615	38	7·5058339	38	7·5164100	38	7·5269900	38	7·5375737	38
12·0	7·6011557	38	7·6117660	38	7·6223800	38	7·6329977	38	7·6436193	38
12·1	7·7074273	37	7·7180751	37	7·7287267	37	7·7393819	37	7·7500409	37
12·2	7·8140731	37	7·8247581	37	7·8354469	37	7·8461393	37	7·8568355	37
12·3	7·9210898	37	7·9318118	37	7·9425374	37	7·9532667	37	7·9639997	37
12·4	8·0284744	36	8·0392330	36	8·0499952	36	8·0607611	36	8·0715306	36
12·5	8·1362237	36	8·1470186	36	8·1578171	36	8·1686192	36	8·1794249	36
12·6	8·2443348	36	8·2551657	36	8·2660001	36	8·2768382	36	8·2876798	36
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12·8	8·4616301	35	8·4725321	35	8·4834376	35	8·4943467	35	8·5052592	35
12·9	8·5708085	35	8·5817456	35	8·5926863	35	8·6036304	35	8·6145780	35
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13·1	8·7902127	34	8·8012193	34	8·8122293	34	8·8232427	34	8·8342596	34
13·2	8·9004329	34	8·9114738	34	8·9225181	34	8·9335658	34	8·9446169	34
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14·7	10·5933288	31	10·6048541	31	10·6163825	31	10·6279139	31	10·6394484	30
14·8	10·7087191	30	10·7202749	30	10·7318337	30	10·7433955	30	10·7549604	30
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15·7	11·7606461	29	11·7724668	29	11·7842903	29	11·7961166	29	11·8079458	29
15·8	11·8789808	28	11·8908299	28	11·9026819	28	11·9145366	28	11·9263943	28
15·9	11·9975993	28	12·0094766	28	12·0213568	28	12·0332399	28	12·0451257	28
16·0	12·1164996	28	12·1284051	28	12·1403134	28	12·1522244	28	12·1641383	28

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7·2321991	39	7·2426752	39	7·2531553	39	7·2636392	39	7·2741270	39	11·6
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7·6542446	38	7·6648736	38	7·6755064	38	7·6861430	37	7·6967833	37	12·0
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7·8675353	37	7·8782388	37	7·8889461	37	7·8996570	37	7·9103716	37	12·2
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17·0	13·3206196	26	13·3327964	26	13·3449759	26	13·3571579	26	13·3693426	26
17·1	13·4425059	26	13·4547090	26	13·4669146	26	13·4791229	26	13·4913338	26
17·2	13·5646538	26	13·5768829	26	13·5891146	26	13·6013489	26	13·6135858	26
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19·0	15·8063410	23	15·8190144	23	15·8316902	23	15·8443683	23	15·8570488	23
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19·7	16·6990771	23	16·7119118	23	16·7247487	23	16·7375878	23	16·7504292	23
19·8	16·8275250	23	16·8403822	22	16·8532417	22	16·8661033	22	16·8789673	22
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20·1	17·2142140	22	17·2271381	22	17·2400645	22	17·2529930	22	17·2659238	22
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20·7	17·9935434	21	18·0065984	21	18·0196556	21	18·0327149	21	18·0457764	21
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20·9	18·2550510	21	18·2681488	21	18·2812487	21	18·2943508	21	18·3074550	21
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13·1382840	27	13·1504211	27	13·1625610	27	13·1747035	27	13·1868486	26	16·8
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13·5035473	26	13·5157634	26	13·5279821	26	13·5402034	26	13·5524273	26	17·1
13·6258253	26	13·6380674	26	13·6503121	26	13·6625594	26	13·6748092	26	17·2
13·7483626	26	13·7606305	26	13·7729010	26	13·7851741	26	13·7974498	26	17·3
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15·4902716	24	15·5028860	24	15·5155027	24	15·5281218	24	15·5407433	24	18·7
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21·0	18·3861246	21	18·3992436	21	18·4123648	21	18·4254881	21	18·4386134	21
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21·2	18·6489062	21	18·6620674	21	18·6752307	21	18·6883961	21	18·7015635	21
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21·9	19·5751973	20	19·5885028	20	19·6018104	20	19·6151200	20	19·6284316	20
22·0	19·7083439	20	19·7216697	20	19·7349975	20	19·7483273	20	19·7616592	20
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22·7	20·6459734	20	20·6594383	20	20·6729052	20	20·6863739	20	20·6998447	20
22·8	20·7807101	19	20·7941945	19	20·8076808	19	20·8211691	19	20·8346593	19
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23·1	21·1860848	19	21·1996272	19	21·2131715	19	21·2267177	19	21·2402659	19
23·2	21·3215951	19	21·3351566	19	21·3487201	19	21·3622855	19	21·3758528	19
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23·8	22·1386449	19	22·1523197	19	22·1659964	19	22·1796750	19	22·1933554	19
23·9	22·2754769	19	22·2891703	19	22·3028656	19	22·3165627	19	22·3302617	19
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25·9	25·0500741	17	25·0641236	17	25·0781748	17	25·0922277	17	25·1062823	17
26·0	25·1906457	17	25·2047122	17	25·2187804	17	25·2328504	17	25·2469220	17

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26·0	25·1906457	17	25·2047122	17	25·2187804	17	25·2328504	17	25·2469220	17
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29·2	29·7754183	15	29·7899981	15	29·8045794	15	29·8191622	15	29·8337465	15
29·3	29·9212842	15	29·9358791	15	29·9504755	15	29·9650734	15	29·9796729	15
29·4	30·0673010	15	30·0819109	15	30·0965224	15	30·1111353	15	30·1257498	15
29·5	30·2134680	15	30·2280929	15	30·2427193	15	30·2573473	15	30·2719767	15
29·6	30·3597847	15	30·3744246	15	30·3890660	15	30·4037088	15	30·4183532	15
29·7	30·5062507	15	30·5209054	15	30·5355617	15	30·5502195	15	30·5648787	15
29·8	30·6528653	15	30·6675350	15	30·6822061	15	30·6968787	15	30·7115527	15
29·9	30·7996282	15	30·8143126	15	30·8289985	15	30·8436859	15	30·8583748	15
30·1	30·9465388	15	30·9612380	15	30·9759386	15	30·9906407	15	31·0053443	15
30·1	31·0935966	15	31·1083105	15	31·1230258	15	31·1377426	15	31·1524608	15
30·2	31·2408011	15	31·2555296	15	31·2702596	15	31·2849910	15	31·2997239	15
30·3	31·3881518	15	31·4028949	15	31·4176395	15	31·4323855	15	31·4471330	15
30·4	31·5356483	15	31·5504059	15	31·5651650	15	31·5799256	15	31·5946876	15
30·5	31·6832900	14	31·6980621	14	31·7128357	14	31·7276107	14	31·7423872	14
30·6	31·8310764	14	31·8458630	14	31·8606510	14	31·8754405	14	31·8902314	14
30·7	31·9790071	14	31·9938081	14	32·0086105	14	32·0234144	14	32·0382197	14
30·8	32·1270816	14	32·1418969	14	32·1567137	14	32·1715319	14	32·1863516	14
30·9	32·2752994	14	32·2901291	14	32·3049601	14	32·3197926	14	32·3346266	14
31·0	32·4236601	14	32·4385040	14	32·4533493	14	32·4681961	14	32·4830442	14

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33·1	35·5714285	13	35·5865615	13	35·6016959	13	35·6168316	13	35·6319686	13
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33·3	35·8743413	13	35·8895009	13	35·9046618	13	35·9198240	13	35·9349876	13
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36·5	40·7908760	12	40·8064397	12	40·8220047	12	40·8375709	12	40·8531383	12
36·6	40·9465679	12	40·9621437	12	40·9777207	12	40·9932989	12	41·0088783	12
36·7	41·1023801	12	41·1179679	12	41·1335569	12	41·1491472	12	41·1647386	12
36·8	41·2583123	12	41·2739121	12	41·2895131	12	41·3051152	12	41·3207186	12
36·9	41·4143640	12	41·4299758	12	41·4455887	12	41·4612029	12	41·4768182	12
37·0	41·5705351	12	41·5861588	12	41·6017837	12	41·6174097	12	41·6330369	12
37·1	41·7268252	12	41·7424608	12	41·7580975	12	41·7737354	12	41·7893745	12
37·2	41·8832340	12	41·8988813	12	41·9145299	12	41·9301797	12	41·9458306	12
37·3	42·0397610	12	42·0554202	12	42·0710806	12	42·0867422	12	42·1024049	12
37·4	42·1964061	12	42·2120771	12	42·2277492	12	42·2434226	12	42·2590971	12
37·5	42·3531688	12	42·3688516	12	42·3845355	12	42·4002206	12	42·4159068	12
37·6	42·5100490	12	42·5257434	12	42·5414391	12	42·5571358	12	42·5728338	12
37·7	42·6670462	12	42·6827523	12	42·6984596	12	42·7141681	12	42·7298777	12
37·8	42·8241601	12	42·8398779	12	42·8555968	12	42·8713170	12	42·8870383	12
37·9	42·9813904	12	42·9971198	12	43·0128504	12	43·0285822	12	43·0443151	12
38·0	43·1387369	12	43·1544779	12	43·1702201	12	43·1859634	12	43·2017079	12
38·1	43·2961991	12	43·3119517	12	43·3277055	12	43·3434604	12	43·3592164	12
38·2	43·4537769	12	43·4695410	12	43·4853063	12	43·5010727	12	43·5168403	12
38·3	43·6114699	11	43·6272455	11	43·6430222	11	43·6588002	11	43·6745792	11
38·4	43·7692777	11	43·7850648	11	43·8008530	11	43·8166424	11	43·8324329	11
38·5	43·9272001	11	43·9429986	11	43·9587983	11	43·9745991	11	43·9904011	11
38·6	44·0852368	11	44·1010468	11	44·1168578	11	44·1326701	11	44·1484834	11
38·7	44·2433875	11	44·2592088	11	44·2750313	11	44·2908549	11	44·3066796	11
38·8	44·4016519	11	44·4174845	11	44·4333184	11	44·4491533	11	44·4649894	11
38·9	44·5600296	11	44·5758736	11	44·5917188	11	44·6075650	11	44·6234124	11
39·0	44·7185205	11	44·7343758	11	44·7502322	11	44·7660897	11	44·7819484	11
39·1	44·8771241	11	44·8929907	11	44·9088584	11	44·9247272	11	44·9405971	11
39·2	45·0358403	11	45·0517181	11	45·0675970	11	45·0834770	11	45·0993582	11
39·3	45·1946686	11	45·2105576	11	45·2264477	11	45·2423390	11	45·2582313	11
39·4	45·3536089	11	45·3695091	11	45·3854104	11	45·4013128	11	45·4172163	11
39·5	45·5126608	11	45·5285722	11	45·5444846	11	45·5603981	11	45·5763128	11
39·6	45·6718241	11	45·6877466	11	45·7036701	11	45·7195948	11	45·7355205	11
39·7	45·8310985	11	45·8470320	11	45·8629666	11	45·8789024	11	45·8948392	11
39·8	45·9904836	11	46·0064282	11	46·0223739	11	46·0383207	11	46·0542686	11
39·9	46·1499792	11	46·1659349	11	46·1818916	11	46·1978494	11	46·2138084	11
40·0	46·3095851	11	46·3255517	11	46·3415194	11	46·3574883	11	46·3734582	11
40·1	46·4693009	11	46·4852785	11	46·5012572	11	46·5172370	11	46·5332179	11
40·2	46·6291263	11	46·6451149	11	46·6611046	11	46·6770953	11	46·6930872	11
40·3	46·7890612	11	46·8050607	11	46·8210612	11	46·8370629	11	46·8530657	11
40·4	46·9491051	11	46·9651155	11	46·9811270	11	46·9971396	11	47·0131532	11
40·5	47·1092579	11	47·1252792	11	47·1413015	11	47·1573250	11	47·1733495	11
40·6	47·2695193	11	47·2855514	11	47·3015846	11	47·3176188	11	47·3336542	11
40·7	47·4298890	11	47·4459319	11	47·4619759	11	47·4780210	11	47·4940671	11
40·8	47·5903667	11	47·6064204	11	47·6224751	11	47·6385310	11	47·6545879	11
40·9	47·7509521	11	47·7670166	11	47·7830821	11	47·7991487	11	47·8152164	11
41·0	47·9116451	11	47·9277203	11	47·9437965	11	47·9598739	11	47·9759523	11

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40·0917601	12	40·1072692	12	40·1227796	12	40·1382912	12	40·1538040	12	36·0
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40·4021747	12	40·4177082	12	40·4332429	12	40·4487789	12	40·4643160	12	36·2
40·5575644	12	40·5731100	12	40·5886569	12	40·6042050	12	40·6197542	12	36·3
40·7130752	12	40·7286330	12	40·7441919	12	40·7597521	12	40·7753134	12	36·4
40·8687069	12	40·8842767	12	40·8998477	12	40·9154199	12	40·9309933	12	36·5
41·0244590	12	41·0400408	12	41·0556238	12	41·0712080	12	41·0867935	12	36·6
41·1803312	12	41·1959250	12	41·2115200	12	41·2271162	12	41·2427136	12	36·7
41·3363232	12	41·3519290	12	41·3675360	12	41·3831441	12	41·3987535	12	36·8
41·4924347	12	41·5080524	12	41·5236713	12	41·5392914	12	41·5549127	12	36·9
41·6486653	12	41·6642949	12	41·6799257	12	41·6955577	12	41·7111909	12	37·0
41·8050148	12	41·8206563	12	41·8362989	12	41·8519427	12	41·8675878	12	37·1
41·9614827	12	41·9771360	12	41·9927905	12	42·0084462	12	42·0241030	12	37·2
42·1180688	12	42·1337339	12	42·1494002	12	42·1650677	12	42·1807363	12	37·3
42·2747728	12	42·2904496	12	42·3061277	12	42·3218069	12	42·3374873	12	37·4
42·4315943	12	42·4472829	12	42·4629726	12	42·4786636	12	42·4943557	12	37·5
42·5885329	12	42·6042333	12	42·6199347	12	42·6356374	12	42·6513412	12	37·6
42·7455885	12	42·7613005	12	42·7770137	12	42·7927280	12	42·8084434	12	37·7
42·9027607	12	42·9184843	12	42·9342091	12	42·9499350	12	42·9656621	12	37·8
43·0600491	12	43·0757844	12	43·0915208	12	43·1072583	12	43·1229970	12	37·9
43·2174535	12	43·2332004	12	43·2489483	12	43·2646974	12	43·2804477	12	38·0
43·3749736	12	43·3907320	12	43·4064915	12	43·4222521	12	43·4380139	12	38·1
43·5326090	12	43·5483789	12	43·5641499	12	43·5799221	11	43·5956954	11	38·2
43·6903594	11	43·7061408	11	43·7219233	11	43·7377069	11	43·7534917	11	38·3
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44·0062042	11	44·0220084	11	44·0378138	11	44·0536203	11	44·0694280	11	38·5
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44·3225055	11	44·3383325	11	44·3541606	11	44·3699899	11	44·3858203	11	38·7
44·4808266	11	44·4966649	11	44·5125044	11	44·5283450	11	44·5441868	11	38·8
44·6392609	11	44·6551106	11	44·6709614	11	44·6868133	11	44·7026663	11	38·9
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45·2741248	11	45·2900194	11	45·3059151	11	45·3218119	11	45·3377099	11	39·3
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45·5922286	11	45·6081455	11	45·6240635	11	45·6399826	11	45·6559028	11	39·5
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46·0702176	11	46·0861677	11	46·1021190	11	46·1180713	11	46·1340247	11	39·8
46·2297684	11	46·2457295	11	46·2616918	11	46·2776551	11	46·2936195	11	39·9
46·3894292	11	46·4054014	11	46·4213746	11	46·4373489	11	46·4533244	11	40·0
46·5491999	11	46·5651830	11	46·5811672	11	46·5971525	11	46·6131389	11	40·1
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47·1893751	11	47·2054017	11	47·2214295	11	47·2374584	11	47·2534883	11	40·5
47·3496906	11	47·3657281	11	47·3817667	11	47·3978064	11	47·4138471	11	40·6
47·5101143	11	47·5261626	11	47·5422120	11	47·5582625	11	47·5743140	11	40·7
47·6706459	11	47·6867050	11	47·7027652	11	47·7188264	11	47·7348887	11	40·8
47·8312852	11	47·8473550	11	47·8634259	11	47·8794979	11	47·8955709	11	40·9
47·9920318	11	48·0081123	11	48·0241939	11	48·0402766	11	48·0563604	11	41·0

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41·0	47·9116451	11	47·9277203	11	47·9437965	11	47·9598739	11	47·9759523	11
41·1	48·0724452	11	48·0885311	11	48·1046181	11	48·1207062	11	48·1367953	11
41·2	48·2333524	11	48·2494490	11	48·2655466	11	48·2816453	11	48·2977451	11
41·3	48·3943662	11	48·4104735	11	48·4265818	11	48·4426911	11	48·4588016	11
41·4	48·5554865	11	48·5716044	11	48·5877233	11	48·6038433	11	48·6199644	11
41·5	48·7167130	11	48·7328414	11	48·7489710	11	48·7651016	11	48·7812332	11
41·6	48·8780453	11	48·8941844	11	48·9103245	11	48·9264657	11	48·9426079	11
41·7	49·0394834	11	49·0556330	11	49·0717836	11	49·0879354	11	49·1040881	11
41·8	49·2010268	11	49·2171870	11	49·2333481	11	49·2495104	11	49·2656737	11
41·9	49·3626754	10	49·3788461	10	49·3950177	10	49·4111905	10	49·4273643	10
42·0	49·5244289	10	49·5406100	10	49·5567922	10	49·5729754	10	49·5891596	10
42·1	49·6862871	10	49·7024786	10	49·7186712	10	49·7348649	10	49·7510596	10
42·2	49·8482496	10	49·8644516	10	49·8806546	10	49·8968587	10	49·9130638	10
42·3	50·0103163	10	50·0265287	10	50·0427421	10	50·0589565	10	50·0751720	10
42·4	50·1724868	10	50·1887096	10	50·2049334	10	50·2211582	10	50·2373841	10
42·5	50·3347611	10	50·3509942	10	50·3672283	10	50·3834635	10	50·3996997	10
42·6	50·4971387	10	50·5133821	10	50·5296266	10	50·5458721	10	50·5621186	10
42·7	50·6596194	10	50·6758732	10	50·6921279	10	50·7083837	10	50·7246406	10
42·8	50·8222031	10	50·8384671	10	50·8547322	10	50·8709982	10	50·8872653	10
42·9	50·9848895	10	51·0011637	10	51·0174390	10	51·0337153	10	51·0499927	10
43·0	51·1476782	10	51·1639627	10	51·1802482	10	51·1965348	10	51·2128223	10
43·1	51·3105692	10	51·3268639	10	51·3431596	10	51·3594563	10	51·3757541	10
43·2	51·4735620	10	51·4898669	10	51·5061728	10	51·5224798	10	51·5387877	10
43·3	51·6366566	10	51·6529717	10	51·6692877	10	51·6856048	10	51·7019229	10
43·4	51·7998527	10	51·8161779	10	51·8325041	10	51·8488313	10	51·8651595	10
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44·0	52·7811467	10	52·7975322	10	52·8139186	10	52·8303061	10	52·8466946	10
44·1	52·9450464	10	52·9614419	10	52·9778383	10	52·9942358	10	53·0106342	10
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44·3	53·2731445	10	53·2895599	10	53·3059762	10	53·3223935	10	53·3388118	10
44·4	53·4373424	10	53·4537677	10	53·4701939	10	53·4866211	10	53·5030493	10
44·5	53·6016392	10	53·6180744	10	53·6345105	10	53·6509475	10	53·6673856	10
44·6	53·7660348	10	53·7824797	10	53·7989257	10	53·8153726	10	53·8318206	10
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50·7408984	10	50·7571573	10	50·7734172	10	50·7896782	10	50·8059401	10	42·7
50·9035335	10	50·9198026	10	50·9360728	10	50·9523440	10	50·9686162	10	42·8
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51·7182420	10	51·7345621	10	51·7508832	10	51·7672054	10	51·7835285	10	43·3
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53·1910828	10	53·2074931	10	53·2239045	10	53·2403169	10	53·2567302	10	44·2
53·3552311	10	53·3716514	10	53·3880727	10	53·4044949	10	53·4209182	10	44·3
53·5194785	10	53·5359087	10	53·5523398	10	53·5687720	10	53·5852051	10	44·4
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46·1	56·2436616	10	56·2602518	10	56·2768429	10	56·2934351	10	56·3100281	10
46·2	56·4096065	10	56·4262062	9	56·4428069	9	56·4594085	9	56·4760111	9
46·3	56·5756465	9	56·5922557	9	56·6088658	9	56·6254770	9	56·6420890	9
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46·6	57·0743345	9	57·0909720	9	57·1076105	9	57·1242500	9	57·1408904	9
46·7	57·2407525	9	57·2573994	9	57·2740474	9	57·2906962	9	57·3073460	9
46·8	57·4072645	9	57·4239208	9	57·4405781	9	57·4572364	9	57·4738956	9
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DEPARTMENT OF APPLIED STATISTICS
(COMPUTING SECTION)
UNIVERSITY OF LONDON, UNIVERSITY COLLEGE

TRACTS FOR COMPUTERS

EDITED BY KARL PEARSON, F.R.S.

No. X (Double Number)

On Quadrature and Cubature
Or
On Methods of Determining Approximately Single and Double Integrals

BY J. O. IRWIN, B.A. CANTAB.

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TRACTS FOR COMPUTERS

No. X

INTRODUCTION

The subject of cubature does not seem, hitherto, to have attracted much attention in spite of the many works that have been written on quadrature; but it may well be of considerable importance, especially in numerical work, owing to the frequent occurrence in analysis of double integrals that do not admit of exact integration.

The principal object of this paper is to deduce cubature formulae from bivariate central difference interpolation formulae; for since these always base the value of a function on its values at the 'nearest' points at which it is known, it was thought that they might give rise to very satisfactory cubature formulae. But, in order that the reader might have readily at his disposal those univariate quadrature formulae which he is most likely to need, and to provide him with central difference quadrature formulae for one variate before proceeding to the discussion of the bivariate case, it has been thought desirable to insert a chapter on quadrature.

While an attempt has been made to give a brief *résumé* of current quadrature formulae no proofs are provided, except where central difference quadrature formulae—believed to be novel—are concerned. Reference to the sources are provided in a bibliographical appendix. Mr J. O. Irwin recently had to determine the approximate values of a number of double integrals and the work led him to a series of central difference cubature formulae, which appear likely to be of service to computers in general. The cubature portion of this tract as well as the central difference quadrature formulae of the first part, are—as far as he is aware—now published for the first time.

KARL PEARSON.

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CONTENTS

ON QUADRATURE AND CUBATURE

OR

ON METHODS OF DETERMINING APPROXIMATELY SINGLE AND DOUBLE INTEGRALS

By J. O. Irwin, B.A. Cantab.

PART I. QUADRATURE

A. Quadrature by Equi-distant Ordinates

Before we proceed to the discussion of any new formulae, it may be well to give some account of the usual methods of quadrature*.

Consider the area of any curve bounded by the ordinates at the points x_0, x_p, the curve itself and the axis of x, and let us suppose the base divided into p equal portions of length h, by the ordinates $z_0, z_1, z_2 \ldots z_p$. Then for obtaining approximations to the area $\int_{x_0}^{x_p} z\,dx \left(= \int_0^{ph} z\,dx, \text{ if our origin is taken at } x_0\right)$ we may use either the extreme ordinates $z_0, z_1, z_2 \ldots z_p$, or the mid-ordinates $z_{\frac{1}{2}}, z_{\frac{3}{2}}, z_{\frac{5}{2}} \ldots z_{p-\frac{1}{2}}$ of the constituent trapezettes.

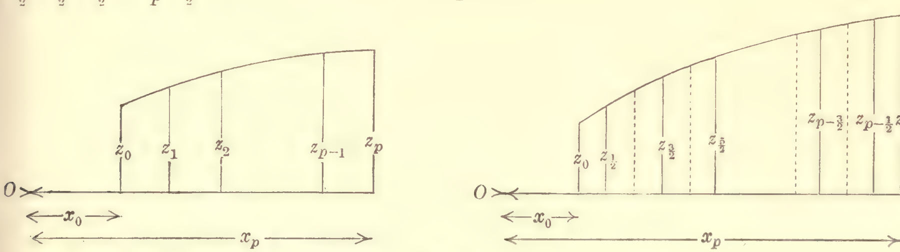

Fig. 1

The fundamental formulae are the Euler-McLaurin formulae

$$\int_{x_0}^{x_p} z\,dx = A_C - \left[\frac{1}{12} h^2 \frac{dz}{dx} - \frac{h^4}{720} \frac{d^3 z}{dx^3} + \frac{h^6}{30240} \frac{d^5 z}{dx^5} - \ldots\right]_{x_0}^{x_p} \quad \ldots\ldots(\alpha),$$

$$\int_{x_0}^{x_p} z\,dx = A_T + \left[\frac{1}{24} h^2 \frac{dz}{dx} - \frac{7h^4}{5760} \frac{d^3 z}{dx^3} + \frac{31h^6}{967680} \frac{d^5 z}{dx^5} - \ldots\right]_{x_0}^{x_p} \ldots\ldots(\beta),$$

* This collection of well-known formulae is taken from *Biometrika*, Vol. I, pp. 273—278. I am indebted to Professor Pearson for permission to use it.

where $$A_C = h\left(\tfrac{1}{2}z_0 + z_1 + z_2 + \ldots + z_{p-1} + \tfrac{1}{2}z_p\right),$$
$$A_T = h\left(z_{\frac{1}{2}} + z_{\frac{3}{2}} + z_{\frac{5}{2}} + \ldots + z_{p-\frac{3}{2}} + z_{p-\frac{1}{2}}\right),$$
and the values of the expressions in square brackets have to be calculated at the points x_0, x_p and the former subtracted from the latter.

For purposes of practical work the differential coefficients in these formulae can be replaced by differences, when we reach

$$\int_{x_0}^{x_p} z\,dx = A_C + h\left(\gamma_1\Delta - \gamma_2\Delta^2 + \gamma_3\Delta^3 - \gamma_4\Delta^4 + \ldots\right)(z_0 + z_p),$$

$$\int_{x_0}^{x_p} z\,dx = A_T - h\left(\gamma_1'\Delta - \gamma_2'\Delta^2 + \gamma_3'\Delta^3 - \gamma_4'\Delta^4 + \ldots\right)(z_{\frac{1}{2}} + z_{p-\frac{1}{2}}).$$

In these formulae Δ operating on z_p and $z_{p-\frac{1}{2}}$ must be taken backwards, i.e. $\Delta z_p = z_{p-1} - z_p$ and $\Delta z_{p-\frac{1}{2}} = z_{p-\frac{3}{2}} - z_{p-\frac{1}{2}}$ while $\Delta z_0 = z_1 - z_0$, $\Delta z_{\frac{1}{2}} = z_{\frac{3}{2}} - z_{\frac{1}{2}}$.

The values of the coefficients γ are:

$\gamma_1 = \cdot 083{,}3333$	$\gamma_1' = \cdot 041{,}6667$
$\gamma_2 = \cdot 041{,}6667$	$\gamma_2' = \cdot 041{,}6667$
$\gamma_3 = \cdot 026{,}3889$	$\gamma_3' = \cdot 038{,}7153$
$\gamma_4 = \cdot 018{,}7500$	$\gamma_4' = \cdot 035{,}7639$
$\gamma_5 = \cdot 014{,}2692$	$\gamma_5' = \cdot 033{,}1918$
$\gamma_6 = \cdot 011{,}3674$	$\gamma_6' = \cdot 030{,}9989$
$\gamma_7 = \cdot 009{,}3565$	$\gamma_7' = \cdot 029{,}1253$
$\gamma_8 = \cdot 007{,}8925$	$\gamma_8' = \cdot 027{,}5110$
$\gamma_9 = \cdot 006{,}7858$	$\gamma_9' = \cdot 026{,}1066$
$\gamma_{10} = \cdot 005{,}9241$	$\gamma_{10}' = \cdot 024{,}8732$
$\gamma_{11} = \cdot 005{,}2367$	$\gamma_{11}' = \cdot 023{,}7807$
$\gamma_{12} = \cdot 004{,}6775$	$\gamma_{12}' = \cdot 022{,}8052$
$\gamma_{13} = \cdot 004{,}2150$	
$\gamma_{14} = \cdot 003{,}8269$	

Professor Pearson has pointed out that these Euler-McLaurin formulae (the correction terms being, as they usually are, small) give equal weight to all the ordinates except the first and last, in the principal portions A_C and A_T of the formulae; a circumstance which is of great importance when the ordinates are observations which are liable to error.

Although they will give the best possible results if we go to the complete system of differences for the ordinates, to do so involves very great labour, and if only three or four γ's are used, they are not the best coefficients by which to multiply the successive differences.

For these reasons the following formulae, in which the number of ordinates used is a multiple of 2, 3, 4, 6 ... etc., are preferable.

Simpson's Rule ($2p$ elements).

$$\int_{x_0}^{x_{2p}} z\,dx = \tfrac{1}{3}h\,\{z_0 + 2(z_2 + z_4 + \ldots + z_{2p-2}) + z_{2p} + 4(z_1 + z_3 + \ldots + z_{2p-1})\} \ldots (\gamma).$$

Newton's Rule ($3p$ elements).

$$\int_{x_0}^{x_{3p}} z\,dx = \tfrac{3}{8}h\,\{z_0 + 3(z_1 + z_2 + z_4 + z_5 + z_7 + z_8 + \ldots) + z_{3p} + 2(z_3 + z_6 + \ldots + z_{3p-3})\} \ldots\ldots (\delta).$$

Boole's Rule ($4p$ elements).

$$\int_{x_0}^{x_{4p}} z\,dx = \tfrac{2}{45}h\,\{7z_0 + 14(z_4 + z_8 + \ldots + z_{4p-4}) + 7z_{4p} + 32(z_1 + z_3 + z_5 + \ldots + z_{4p-1}) + 12(z_2 + z_6 + z_{10} + \ldots + z_{4p-2})\} \ldots\ldots (\epsilon).$$

Weddle's Rule ($6p$ elements).

$$\int_{x_0}^{x_{6p}} z\,dx = \frac{3h}{10}\{z_0 + z_2 + z_4 + z_8 + z_{10} + \ldots + z_{6p-2} + z_{6p} + 2(z_6 + z_{12} + \ldots + z_{6p-6}) + 5(z_1 + z_5 + z_7 + z_{11} + \ldots + z_{6p-1}) + 6(z_3 + z_9 + z_{15} + \ldots + z_{6p-3})\} \ldots\ldots (\zeta).$$

These formulae, whose exactness is in the order in which they are written down, also have their disadvantages. It is not always possible to select beforehand the number of elements to be used, and this number may not fit in with any of the above formulae; secondly, all the ordinates are not equally weighted which is a disadvantage if they are observational quantities subject to errors. Formulae (ϵ) and (ζ), particularly (ζ), have been found to give very good results for continuous mathematical functions, but they are not so good for observational data.

For this purpose the following formulae of Dr Sheppard's are better. The corrective terms are so chosen, as to make the first term, in the Euler-McLaurin formulae, which they neglect, as small as possible*.

Case (i). *Bounding ordinates or chordal area known.*

(*a*) *One Difference:*

$$\text{Area} = A_C + \frac{1}{12}\frac{p}{p-1}\{(z_1 - z_0) - (z_p - z_{p-1})\}\,h \ldots\ldots (\eta).$$

(*b*) *Two Differences:*

$$\text{Area} = A_C + \frac{1}{120}\frac{p(15p-26)}{(p-1)(p-2)}\{(z_1 - z_0) - (z_p - z_{p-1})\}\,h - \frac{1}{120}\frac{p(5p-6)}{(p-2)(p-3)}\{(z_2 - z_1) - (z_{p-1} - z_{p-2})\}\,h \ldots (\theta).$$

* See *Proceedings of the London Mathematical Society*, Vol. XXXII, p. 270.

(c) *Three Differences:*

$$\text{Area} = A_C + \frac{1}{5040}\frac{p(763p^2 - 3444p + 3636)}{(p-1)(p-2)(p-3)}\{(z_1 - z_0) - (z_p - z_{p-1})\}h$$
$$-\frac{1}{1260}\frac{p(119p^2 - 504p + 432)}{(p-2)(p-3)(p-4)}\{(z_2 - z_1) - (z_{p-1} - z_{p-2})\}h$$
$$+\frac{1}{5040}\frac{p(113p^2 - 462p + 360)}{(p-3)(p-4)(p-5)}\{(z_3 - z_2) - (z_{p-2} - z_{p-3})\}h^* \quad \ldots\ldots\ldots\ldots(\iota).$$

Case (ii). *Mid-ordinates or tangential area known.*

(a) *One Difference:*

$$\text{Area} = A_T - \frac{1}{24}\frac{p}{(p-2)}\{(z_{\frac{3}{2}} - z_{\frac{1}{2}}) - (z_{p-\frac{1}{2}} - z_{p-\frac{3}{2}})\}h \quad \ldots\ldots\ldots(\kappa).$$

(b) *Two Differences:*

$$\text{Area} = A_T - \frac{1}{960}\frac{p(80p - 177)}{(p-2)(p-3)}\{(z_{\frac{3}{2}} - z_{\frac{1}{2}}) - (z_{p-\frac{1}{2}} - z_{p-\frac{3}{2}})\}h$$
$$+\frac{1}{960}\frac{p(40p - 57)}{(p-3)(p-4)}\{(z_{\frac{5}{2}} - z_{\frac{3}{2}}) - (z_{p-\frac{3}{2}} - z_{p-\frac{5}{2}})\}h \quad \ldots(\lambda).$$

This formula, Professor Pearson has pointed out, has many advantages. It is more exact than (κ) and sufficiently so for practical purposes, though not quite as exact as (μ). Further, all the ordinates that occur in the expression A_T are equally weighted, which is not the case with A_C, where the end ordinates only have half-weight. It may be written

$$\text{Area} = A_T - P\{(z_{\frac{3}{2}} - z_{\frac{1}{2}}) - (z_{p-\frac{1}{2}} - z_{p-\frac{3}{2}})\}h + Q\{(z_{\frac{5}{2}} - z_{\frac{3}{2}}) - (z_{p-\frac{3}{2}} - z_{p-\frac{5}{2}})\}h,$$

where the following are the values of P and Q:

p	P	Q	p	P	Q
8	·128,6111	·109,5833	15	·102,4639	·064,2756
9	·121,2054	·094,6875	16	·101,0073	·062,2863
10	·115,8854	·085,0694	17	·099,7569	·060,6170
11	·111,8779	·078,3668	18	·098,6719	·059,1964
12	·108,7500	·073,4375	19	·097,7214	·057,9731
13	·106,2405	·069,6644	20	·096,8818	·056,9087
14	·104,1825	·066,6856			

This formula gives results which are, as a rule, better than Simpson's, and only those ordinates are weighted which occur in the corrective terms.

* The coefficients in this formula have been tabulated, for values of p from 6 to 100, by P. F. Everitt. See *Biometrika*, Vol. XII, p. 283.

(*c*) *Three Differences:*

$$\text{Area} = A_T - \frac{1}{80640}\frac{p(9842p^2 - 53970p + 70407)}{(p-2)(p-3)(p-4)}\{(z_{\frac{3}{2}} - z_{\frac{1}{2}}) - (z_{p-\frac{1}{2}} - z_{p-\frac{3}{2}})\}\,h$$
$$+ \frac{1}{40320}\frac{p(4802p^2 - 23016p + 22905)}{(p-3)(p-4)(p-5)}\{(z_{\frac{5}{2}} - z_{\frac{3}{2}}) - (z_{p-\frac{3}{2}} - z_{p-\frac{5}{2}})\}\,h$$
$$- \frac{1}{80640}\frac{p(3122p^2 - 12222p + 10935)}{(p-4)(p-5)(p-6)}\{(z_{\frac{7}{2}} - z_{\frac{5}{2}}) - (z_{p-\frac{5}{2}} - z_{p-\frac{7}{2}})\}\,h \quad \text{............}(\mu).$$

This is rather complicated to use in practice, though with the aid of a calculating machine, the coefficients might easily be tabulated.

Case (iii). *Mid-ordinates and two extreme ordinates known.*

(*a*) *One Difference:*

$$\text{Area} = A_T - \frac{1}{12}\frac{2p}{2p-1}\{(z_{\frac{1}{2}} - z_0) - (z_p - z_{p-\frac{1}{2}})\}\,h \quad \text{............}(\nu).$$

(*b*) *Two Differences:*

$$\text{Area} = A_T - \frac{1}{180}\frac{2p(40p-57)}{(2p-1)(2p-3)}\{(z_{\frac{1}{2}} - z_0) - (z_p - z_{p-\frac{1}{2}})\}\,h$$
$$+ \frac{1}{180}\frac{2p(5p-6)}{(2p-2)(2p-3)}\{(z_{\frac{3}{2}} - z_{\frac{1}{2}}) - (z_{p-\frac{1}{2}} - z_{p-\frac{3}{2}})\}\,h \quad \text{...}(\xi).$$

Case (iv). *Bounding ordinates, with the two mid-ordinates only, of the terminal elements.*

(*a*) *One Difference:*

$$\text{Area} = A_C + \frac{1}{6}\frac{2p}{2p-1}\{(z_{\frac{1}{2}} - z_0) - (z_p - z_{p-\frac{1}{2}})\}\,h \quad \text{............}(o).$$

(*b*) *Two Differences:*

$$\text{Area} = A_C + \frac{1}{120}\frac{2p(30p-29)}{(2p-1)(p-1)}\{(z_{\frac{1}{2}} - z_0) - (z_p - z_{p-\frac{1}{2}})\}\,h$$
$$- \frac{1}{120}\frac{2p(10p-9)}{(2p-3)(p-1)}\{(z_1 - z_{\frac{1}{2}}) - (z_{p-\frac{1}{2}} - z_{p-1})\}\,h \quad \text{...}(\pi).$$

It has been pointed out that, for fairly large values of p this is not very divergent from

$$\text{Area} = A_C + \tfrac{1}{4}\{(z_{\frac{1}{2}} - z_0) - (z_p - z_{p-\frac{1}{2}})\}\,h - \tfrac{1}{12}\{(z_1 - z_{\frac{1}{2}}) - (z_{p-\frac{1}{2}} - z_{p-1})\}\,h \quad \text{............}(\rho),$$

which may be obtained directly by a double application of Simpson's formula, and is somewhat more accurate than the latter.

The following table shows the results given by the above formulae, when used to calculate $\int_0^1 \frac{dx}{1+x} = \cdot 693,147,18$. This integral is rather a favourable one on which to test an individual formula, but it may perhaps serve to compare the accuracy obtained by different formulae. Either 12 or 13 ordinates have been used in every case.

	Divergence		Divergence
(α) with four differences	$+\cdot 000,000,25$	(κ)	$-\cdot 000,014,93$
(β) „ „ „	$-\cdot 000,000,59$	(λ)	$-\cdot 000,001,26$
(γ)	$+\cdot 000,001,48$	(μ)	$-\cdot 000,000,12$
(δ)	$+\cdot 000,003,28$	(ν)	$-\cdot 000,003,91$
(ϵ)	$+\cdot 000,000,07$	(ξ)	$-\cdot 000,000,14$
(ζ)	$+\cdot 000,000,04$	(o)	$+\cdot 000,008,12$
(η)	$+\cdot 000,014,59$	(π)	$+\cdot 000,000,22$
(θ)	$+\cdot 000,000,93$	(ρ)	$+\cdot 000,001,27$
(ι)	$+\cdot 000,000,07$		

Also $A_C = \cdot 693,580,83$. $\Delta = \cdot 000,433,65$.

$A_T = \cdot 692,930,49$. $\Delta = -\cdot 000,216,69$*.

B. The Gaussian Method of Quadrature

No discussion of quadrature would be complete without some reference to Gauss' method, in which the ordinates are no longer taken at equal intervals, but at such intervals as make the approximation obtained as close as possible.

Suppose it be required to approximate to $\int_{-1}^1 f(x)\,dx$. Let $P_n(x)$ be Legendre's polynomial of the nth order. Then

$$P_n(x) = \frac{1}{2^n \lfloor n} \frac{d^n}{dx^n}(x^2-1)^n$$

and
$$\int_{-1}^1 x^r P_n(x)\,dx = 0, \text{ for } r = 0, 1, 2 \ldots n-1\dagger.$$

Now suppose $f(x)$ to be a polynomial of degree not greater than $2n-1$. Then by division we have $f(x) = P_n(x)\,Q(x) + R(x)$, where $Q(x)$, $R(x)$ are of degree not greater than $n-1$.

$$\therefore \int_{-1}^1 f(x)\,dx = \int_{-1}^1 P_n(x)\,Q(x)\,dx + \int_{-1}^1 R(x)\,dx;$$

but
$$\int_{-1}^1 P_n(x)\,Q(x)\,dx = 0,$$

$$\therefore \int_{-1}^1 f(x)\,dx = \int_{-1}^1 R(x)\,dx.$$

* See Pearson, 'On the systematic fitting of curves to observations and measurements,' *Biometrika*, Vol. I, p. 265.

† See Whittaker and Watson's *Modern Analysis*, pp. 296—300.

Now $f(x) = R(x)$ for the values $x = a_1, a_2 \ldots a_n$ which are the roots of $P_n(x) = 0$.

But by Lagrange's interpolation formula*

$$R(x) = \sum_{r=1}^{r=n} A_r R(a_r),$$

where
$$A_r = \frac{P_n(x)}{P'_n(a_r)(x - a_r)},$$

whence we deduce the quadrature formula

$$\int_{-1}^{1} f(x)\,dx = \sum_{r=1}^{r=n} B_r f(a_r),$$

where
$$B_r = \int_{-1}^{1} \frac{P_n(x)\,dx}{P'_n(a_r)(x - a_r)},$$

or we have a quadrature formula involving n ordinates only, which is exact for a function of degree not greater than $2n - 1$, whereas, if we had taken the ordinates at equal intervals we should have required $2n$ to obtain the integral exactly.

Gauss actually considered the integral $\int_0^1 f(x)\,dx$, which only requires a slight modification of the above argument, and his result for the case of seven ordinates is as follows: $\int_0^1 f(x)\,dx = \sum_{r=1}^{r=7} R_r f(a_r)$, where we have the following values for the R's and a's†.

r	a_r	R_r
1	·02544 6043829	·06474 2483084
2	·12923 4407200	·13985 2695745
3	·29707 7424311	·19091 5025253
4	·5	·20897 9519837
5	·70292 2575689	·19091 5025253
6	·87076 5592800	·13985 2695745
7	·97455 3956171	·06474 2483084

For the integral $\int_a^b f(x)\,dx$ the formula becomes

$$\int_a^b f(x)\,dx = \sum_{r=1}^{r=7} (b-a) f\{a + (b-a)\,a_r\} R_r,$$

a_r and R_r having the above values.

* See *Tracts for Computers*, II, p. 17.

† Gauss, *Werke*, III, p. 195, Göttingen, 1866. Gauss gives these values to 15 decimals, and we have here given them to 12. Naturally it will not usually be necessary to use all 12.

	0		1		2		3		4	
16·0	12·1164996	28	12·1284051	28	12·1403133	28	12·1522244	28	12·1641383	28
16·1	12·2356801	28	12·2476134	28	12·2595496	28	12·2714885	28	12·2834303	28
16·2	12·3551388	28	12·3670999	28	12·3790638	28	12·3910304	28	12·4029998	28
16·3	12·4748741	27	12·4868627	27	12·4988542	27	12·5108483	27	12·5228452	27
16·4	12·5948841	27	12·6069002	27	12·6189190	27	12·6309405	27	12·6429647	27
16·5	12·7151672	27	12·7272105	27	12·7392565	27	12·7513052	27	12·7633565	27
16·6	12·8357217	27	12·8477920	27	12·8598650	27	12·8719407	27	12·8840191	27
16·7	12·9565458	27	12·9686430	27	12·9807428	27	12·9928454	27	13·0049506	27
16·8	13·0776379	27	13·0897618	27	13·1018884	27	13·1140176	27	13·1261494	27
16·9	13·1989964	26	13·2111468	26	13·2232999	26	13·2354556	26	13·2476140	26
17·0	13·3206196	26	13·3327964	26	13·3449759	26	13·3571579	26	13·3693426	26
17·1	13·4425059	26	13·4547090	26	13·4669146	26	13·4791229	26	13·4913338	26
17·2	13·5646538	26	13·5768829	26	13·5891146	26	13·6013489	26	13·6135858	26
17·3	13·6870617	26	13·6993167	26	13·7115743	26	13·7238345	26	13·7360973	26
17·4	13·8097280	26	13·8220088	26	13·8342921	26	13·8465780	26	13·8588665	26
17·5	13·9326512	26	13·9449576	26	13·9572665	26	13·9695780	25	13·9818920	25
17·6	14·0558298	25	14·0681616	25	14·0804960	25	14·0928329	25	14·1051724	25
17·7	14·1792623	25	14·1916194	25	14·2039791	25	14·2163413	25	14·2287060	25
17·8	14·3029472	25	14·3153295	25	14·3277143	25	14·3401017	25	14·3524915	25
17·9	14·4268831	25	14·4392904	25	14·4517003	25	14·4641126	25	14·4765274	25
18·0	14·5510685	25	14·5635007	25	14·5759354	25	14·5883726	25	14·6008122	25
18·1	14·6755020	25	14·6879590	25	14·7004184	25	14·7128803	25	14·7253446	25
18·2	14·8001823	25	14·8126638	25	14·8251478	25	14·8376342	24	14·8501231	24
18·3	14·9251078	24	14·9376138	24	14·9501222	24	14·9626331	24	14·9751464	24
18·4	15·0502772	24	15·0628075	24	15·0753403	24	15·0878754	24	15·1004130	24
18·5	15·1756892	24	15·1882437	24	15·2008006	24	15·2133599	24	15·2259216	24
18·6	15·3013424	24	15·3139210	24	15·3265019	24	15·3390852	24	15·3516710	24
18·7	15·4272355	24	15·4398380	24	15·4524428	24	15·4650500	24	15·4776596	24
18·8	15·5533672	24	15·5659934	24	15·5786220	24	15·5912530	24	15·6038864	24
18·9	15·6797361	24	15·6923860	24	15·7050383	24	15·7176929	24	15·7303498	24
19·0	15·8063410	23	15·8190144	23	15·8316902	23	15·8443683	23	15·8570488	23
19·1	15·9331806	23	15·9458774	23	15·9585766	23	15·9712781	23	15·9839819	23
19·2	16·0602536	23	16·0729737	23	16·0856962	23	16·0984209	23	16·1111479	23
19·3	16·1875589	23	16·2003021	23	16·2130477	23	16·2257955	23	16·2385457	23
19·4	16·3150950	23	16·3278613	23	16·3406299	23	16·3534007	23	16·3661739	23
19·5	16·4428610	23	16·4556501	23	16·4684416	23	16·4812353	23	16·4940314	23
19·6	16·5708554	23	16·5836673	23	16·5964816	23	16·6092981	23	16·6221169	23
19·7	16·6990771	23	16·7119118	23	16·7247487	23	16·7375878	23	16·7504292	23
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19·9	16·9561979	22	16·9690775	22	16·9819594	22	16·9948435	22	17·0077298	22
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20·1	17·2142140	22	17·2271381	22	17·2400645	22	17·2529930	22	17·2659238	22
20·2	17·3435549	22	17·3565011	22	17·3694495	22	17·3824002	22	17·3953530	22
20·3	17·4731162	22	17·4860844	22	17·4990548	22	17·5120274	22	17·5250022	22
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20·7	17·9935434	21	18·0065984	21	18·0196556	21	18·0327149	21	18·0457764	21
20·8	18·1241902	21	18·1372667	21	18·1503453	21	18·1634261	21	18·1765089	21
20·9	18·2550510	21	18·2681488	21	18·2812487	21	18·2943508	21	18·3074550	21
21·0	18·3861246	21	18·3992436	21	18·4123648	21	18·4254881	21	18·4386134	21

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12·1760549	28	12·1879744	28	12·1998966	28	12·2118216	28	12·2237495	28	16·0
12·2953747	28	12·3073220	28	12·3192721	28	12·3312249	28	12·3431805	28	16·1
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13·2597750	26	13·2719386	26	13·2841049	26	13·2962739	26	13·3084454	26	16·9
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14·2410733	25	14·2534430	25	14·2658153	25	14·2781901	25	14·2905674	25	17·7
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16·2512982	23	16·2640529	23	16·2768100	23	16·2895694	23	16·3023311	23	19·3
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16·5068297	23	16·5196303	23	16·5324331	23	16·5452383	23	16·5580457	23	19·5
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16·7632729	23	16·7761188	23	16·7889670	23	16·8018174	23	16·8146701	23	19·7
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17·0206184	22	17·0335092	22	17·0464022	22	17·0592974	22	17·0721949	22	19·9
17·1496265	22	17·1625396	22	17·1754549	22	17·1883724	22	17·2012921	22	20·0
17·2788568	22	17·2917920	22	17·3047294	22	17·3176690	22	17·3306109	22	20·1
17·4083080	22	17·4212653	22	17·4342247	22	17·4471863	22	17·4601502	22	20·2
17·5379791	22	17·5509583	22	17·5639396	22	17·5769232	22	17·5899089	22	20·3
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17·7979764	22	17·8109991	22	17·8240240	22	17·8370510	22	17·8500801	22	20·5
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18·1895940	21	18·2026811	21	18·2157704	21	18·2288618	21	18·2419553	21	20·8
18·3205613	21	18·3336697	21	18·3467802	21	18·3598929	21	18·3730077	21	20·9
18·4517409	21	18·4648705	21	18·4780022	21	18·4911361	21	18·5042720	21	21·0

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21·0	18·3861246	21	18·3992436	21	18·4123648	21	18·4254881	21	18·4386134	21
21·1	18·5174100	21	18·5305502	21	18·5436925	21	18·5568368	21	18·5699833	21
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21·3	18·7806122	21	18·7937943	21	18·8069785	21	18·8201648	21	18·8333531	21
21·4	18·9125269	21	18·9257299	21	18·9389348	21	18·9521419	21	18·9653511	21
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21·6	19·1769787	21	19·1902229	21	19·2034692	21	19·2167176	21	19·2299681	21
21·7	19·3095137	20	19·3227785	20	19·3360453	20	19·3493142	20	19·3625851	20
21·8	19·4422536	20	19·4555388	20	19·4688260	20	19·4821153	20	19·4954067	20
21·9	19·5751973	20	19·5885028	20	19·6018104	20	19·6151200	20	19·6284316	20
22·0	19·7083439	20	19·7216697	20	19·7349975	20	19·7483273	20	19·7616592	20
22·1	19·8416925	20	19·8550384	20	19·8683864	20	19·8817363	20	19·8950883	20
22·2	19·9752421	20	19·9886081	20	20·0019761	20	20·0153461	20	20·0287180	20
22·3	20·1089918	20	20·1223778	20	20·1357657	20	20·1491556	20	20·1625475	20
22·4	20·2429407	20	20·2563465	20	20·2697543	20	20·2831641	20	20·2965758	20
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22·7	20·6459734	20	20·6594383	20	20·6729052	20	20·6863739	20	20·6998447	20
22·8	20·7807101	19	20·7941945	19	20·8076808	19	20·8211691	19	20·8346593	19
22·9	20·9156414	19	20·9291452	19	20·9426510	19	20·9561586	19	20·9696683	19
23·0	21·0507666	19	21·0642897	19	21·0778148	19	21·0913418	19	21·1048708	19
23·1	21·1860848	19	21·1996272	19	21·2131715	19	21·2267177	19	21·2402659	19
23·2	21·3215951	19	21·3351566	19	21·3487201	19	21·3622855	19	21·3758528	19
23·3	21·4572967	19	21·4708773	19	21·4844599	19	21·4980443	19	21·5116307	19
23·4	21·5931887	19	21·6067884	19	21·6203899	19	21·6339934	19	21·6475987	19
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23·8	22·1386449	19	22·1523197	19	22·1659964	19	22·1796750	19	22·1933554	19
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24·3	22·8246526	18	22·8384196	18	22·8521884	18	22·8659591	18	22·8797315	18
24·4	22·9624046	18	22·9761898	18	22·9899768	18	23·0037656	18	23·0175563	18
24·5	23·1003383	18	23·1141416	18	23·1279467	18	23·1417537	18	23·1555624	18
24·6	23·2384529	18	23·2522743	18	23·2660974	18	23·2799224	18	23·2937492	18
24·7	23·3767477	18	23·3905870	18	23·4044282	18	23·4182711	18	23·4321158	18
24·8	23·5152219	18	23·5290791	18	23·5429382	18	23·5567990	18	23·5706616	18
24·9	23·6538748	18	23·6677499	18	23·6816268	18	23·6955054	18	23·7093858	18
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25·1	23·9317138	18	23·9456243	18	23·9595366	18	23·9734507	18	23·9873665	18
25·2	24·0708984	18	24·0848266	18	24·0987565	18	24·1126881	18	24·1266216	18
25·3	24·2102589	18	24·2242046	18	24·2381520	18	24·2521012	17	24·2660521	17
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22·0702988	19	22·0839643	19	22·0976316	19	22·1113009	19	22·1249720	19	23·7
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22·4810725	18	22·4947937	18	22·5085167	18	22·5222415	18	22·5359682	18	24·0
22·6183670	18	22·6321065	18	22·6458479	18	22·6595911	18	22·6733362	18	24·1
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23·1693730	18	23·1831854	18	23·1969995	18	23·2108155	18	23·2246333	18	24·5
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23·7232680	18	23·7371520	18	23·7510378	18	23·7649253	18	23·7788146	18	24·9
23·8621876	18	23·8760893	18	23·8899928	18	23·9038980	18	23·9178050	18	25·0
24·0012841	18	24·0152034	18	24·0291245	18	24·0430474	18	24·0569720	18	25·1
24·1405567	18	24·1544936	18	24·1684323	18	24·1823728	18	24·1963149	18	25·2
24·2800048	17	24·2939592	17	24·3079154	17	24·3218733	17	24·3358330	17	25·3
24·4196276	17	24·4335995	17	24·4475731	17	24·4615484	17	24·4755255	17	25·4
24·5594245	17	24·5734137	17	24·5874047	17	24·6013974	17	24·6153918	17	25·5
24·6993947	17	24·7134012	17	24·7274095	17	24·7414195	17	24·7554312	17	25·6
24·8395376	17	24·8535613	17	24·8675868	17	24·8816140	17	24·8956429	17	25·7
24·9798524	17	24·9938933	17	25·0079360	17	25·0219803	17	25·0360264	17	25·8
25·1203386	17	25·1343966	17	25·1484563	17	25·1625177	17	25·1765809	17	25·9
25·2609953	17	25·2750704	17	25·2891471	17	25·3032256	17	25·3173057	17	26·0

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26·0	25·1906457	17	25·2047122	17	25·2187804	17	25·2328504	17	25·2469220	17
26·1	25·3313875	17	25·3454710	17	25·3595563	17	25·3736432	17	25·3877318	17
26·2	25·4722990	17	25·4863994	17	25·5005016	17	25·5146054	17	25·5287109	17
26·3	25·6133794	17	25·6274967	17	25·6416157	17	25·6557364	17	25·6698587	17
26·4	25·7546281	17	25·7687622	17	25·7828980	17	25·7970355	17	25·8111746	17
26·5	25·8960445	17	25·9101954	17	25·9243479	17	25·9385020	17	25·9526579	17
26·6	26·0376279	17	26·0517954	17	26·0659646	17	26·0801354	17	26·0943079	17
26·7	26·1793777	17	26·1935618	17	26·2077476	17	26·2219350	17	26·2361241	17
26·8	26·3212933	17	26·3354939	17	26·3496962	17	26·3639001	16	26·3781057	16
26·9	26·4633739	16	26·4775910	16	26·4918098	16	26·5060302	16	26·5202522	16
27·0	26·6056190	16	26·6198526	16	26·6340877	16	26·6483245	16	26·6625630	16
27·1	26·7480280	16	26·7622779	16	26·7765294	16	26·7907826	16	26·8050374	16
27·2	26·8906003	16	26·9048664	16	26·9191342	16	26·9334037	16	26·9476747	16
27·3	27·0333351	16	27·0476176	16	27·0619016	16	27·0761872	16	27·0904745	16
27·4	27·1762321	16	27·1905306	16	27·2048308	16	27·2191326	16	27·2334361	16
27·5	27·3192904	16	27·3336051	16	27·3479214	16	27·3622393	16	27·3765588	16
27·6	27·4625096	16	27·4768403	16	27·4911727	16	27·5055066	16	27·5198421	16
27·7	27·6058890	16	27·6202357	16	27·6345840	16	27·6489340	16	27·6632855	16
27·8	27·7494281	16	27·7637907	16	27·7781550	16	27·7925208	16	27·8068882	16
27·9	27·8931262	16	27·9075047	16	27·9218848	16	27·9362665	16	27·9506498	16
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28·2	28·3251692	16	28·3395950	16	28·3540224	16	28·3684513	16	28·3828818	16
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28·5	28·7586231	16	28·7730957	16	28·7875698	16	28·8020455	15	28·8165227	15
28·6	28·9034187	15	28·9179067	15	28·9323963	15	28·9468875	15	28·9613802	15
28·7	29·0483688	15	29·0628722	15	29·0773773	15	29·0918838	15	29·1063919	15
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28·9	29·3387304	15	29·3532646	15	29·3678003	15	29·3823375	15	29·3968762	15
29·0	29·4841408	15	29·4986903	15	29·5132412	15	29·5277937	15	29·5423477	15
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29·2	29·7754183	15	29·7899981	15	29·8045794	15	29·8191622	15	29·8337465	15
29·3	29·9212842	15	29·9358791	15	29·9504755	15	29·9650734	15	29·9796729	15
29·4	30·0673010	15	30·0819109	15	30·0965224	15	30·1111353	15	30·1257498	15
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30·8	32·1270816	14	32·1418969	14	32·1567137	14	32·1715319	14	32·1863516	14
30·9	32·2752994	14	32·2901291	14	32·3049601	14	32·3197926	14	32·3346266	14
31·0	32·4236601	14	32·4385040	14	32·4533493	14	32·4681961	14	32·4830442	14

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26·8192938	16	26·8335518	16	26·8478115	16	26·8620728	16	26·8763357	16	27·1
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31·1	32·5721631	14	32·5870212	14	32·6018808	14	32·6167417	14	32·6316041	14
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31·7	33·4661455	14	33·4810879	14	33·4960317	14	33·5109769	14	33·5259235	14
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34·2	37·2439566	13	37·2592337	13	37·2745121	13	37·2897918	13	37·3050728	13
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32·9440406	14	32·9589340	14	32·9738289	14	32·9887251	14	33·0036228	14	31·3
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33·9899554	14	34·0049463	14	34·0199385	14	34·0349322	14	34·0499271	14	32·0
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35·1935372	13	35·2086368	13	35·2237377	13	35·2388400	13	35·2539436	13	32·8
35·3445934	13	35·3597064	13	35·3748207	13	35·3899364	13	35·4050534	13	32·9
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45·0	54·4245993	10	54·4410835	10	54·4575687	10	54·4740548	10	54·4905419	10
45·1	54·5894850	10	54·6059789	10	54·6224738	10	54·6389697	10	54·6554666	10
45·2	54·7544681	10	54·7709717	10	54·7874763	10	54·8039819	10	54·8204885	10
45·3	54·9195483	10	54·9360616	10	54·9525759	10	54·9690912	10	54·9856075	10
45·4	55·0847254	10	55·1012484	10	55·1177724	10	55·1342974	10	55·1508233	10
45·5	55·2499993	10	55·2665320	10	55·2830656	10	55·2996002	10	55·3161358	10
45·6	55·4153696	10	55·4319120	10	55·4484553	10	55·4649995	10	55·4815447	10
45·7	55·5808363	10	55·5973882	10	55·6139411	10	55·6304950	10	55·6470498	10
45·8	55·7463990	10	55·7629606	10	55·7795231	10	55·7960865	10	55·8126510	10
45·9	55·9120576	10	55·9286287	10	55·9452008	10	55·9617738	10	55·9783478	10
46·0	56·0778119	10	56·0943925	10	56·1109742	10	56·1275568	10	56·1441403	10

5		6		7		8		9		
47·9920318	11	48·0081123	11	48·0241939	11	48·0402766	11	48·0563604	11	41·0
48·1528855	11	48·1689767	11	48·1850690	11	48·2011624	11	48·2172569	11	41·1
48·3138460	11	48·3299479	11	48·3460509	11	48·3621549	11	48·3782600	11	41·2
48·4749131	11	48·4910256	11	48·5071393	11	48·5232539	11	48·5393697	11	41·3
48·6360865	11	48·6522097	11	48·6683339	11	48·6844592	11	48·7005855	11	41·4
48·7973659	11	48·8134997	11	48·8296345	11	48·8457704	11	48·8619073	11	41·5
48·9587512	11	48·9748955	11	48·9910409	11	49·0071873	11	49·0233348	11	41·6
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49·4435391	10	49·4597150	10	49·4758919	10	49·4920698	10	49·5082489	10	41·9
49·6053449	10	49·6215313	10	49·6377187	10	49·6539071	10	49·6700965	10	42·0
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49·9292699	10	49·9454771	10	49·9616854	10	49·9778946	10	49·9941049	10	42·2
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50·4159370	10	50·4321752	10	50·4484145	10	50·4646549	10	50·4808963	10	42·5
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50·7408984	10	50·7571573	10	50·7734172	10	50·7896782	10	50·8059401	10	42·7
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52·5354843	10	52·5518548	10	52·5682263	10	52·5845988	10	52·6009723	10	43·8
52·6992342	10	52·7156147	10	52·7319962	10	52·7483787	10	52·7647622	10	43·9
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46·0	56·0778119	10	56·0943925	10	56·1109742	10	56·1275568	10	56·1441403	10
46·1	56·2436616	10	56·2602518	10	56·2768429	10	56·2934351	10	56·3100281	10
46·2	56·4096065	10	56·4262062	9	56·4428069	9	56·4594085	9	56·4760111	9
46·3	56·5756465	9	56·5922557	9	56·6088658	9	56·6254770	9	56·6420890	9
46·4	56·7417813	9	56·7583999	9	56·7750196	9	56·7916401	9	56·8082617	9
46·5	56·9080107	9	56·9246388	9	56·9412679	9	56·9578979	9	56·9745289	9
46·6	57·0743345	9	57·0909720	9	57·1076105	9	57·1242500	9	57·1408904	9
46·7	57·2407525	9	57·2573994	9	57·2740474	9	57·2906962	9	57·3073460	9
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Or
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BY J. O. IRWIN, B.A. CANTAB.

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TRACTS FOR COMPUTERS

No. X

INTRODUCTION

The subject of cubature does not seem, hitherto, to have attracted much attention in spite of the many works that have been written on quadrature; but it may well be of considerable importance, especially in numerical work, owing to the frequent occurrence in analysis of double integrals that do not admit of exact integration.

The principal object of this paper is to deduce cubature formulae from bivariate central difference interpolation formulae; for since these always base the value of a function on its values at the 'nearest' points at which it is known, it was thought that they might give rise to very satisfactory cubature formulae. But, in order that the reader might have readily at his disposal those univariate quadrature formulae which he is most likely to need, and to provide him with central difference quadrature formulae for one variate before proceeding to the discussion of the bivariate case, it has been thought desirable to insert a chapter on quadrature.

While an attempt has been made to give a brief *résumé* of current quadrature formulae no proofs are provided, except where central difference quadrature formulae—believed to be novel—are concerned. Reference to the sources are provided in a bibliographical appendix. Mr J. O. Irwin recently had to determine the approximate values of a number of double integrals and the work led him to a series of central difference cubature formulae, which appear likely to be of service to computers in general. The cubature portion of this tract as well as the central difference quadrature formulae of the first part, are—as far as he is aware—now published for the first time.

KARL PEARSON.

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CONTENTS

ON QUADRATURE AND CUBATURE

OR

ON METHODS OF DETERMINING APPROXIMATELY SINGLE AND DOUBLE INTEGRALS

By J. O. Irwin, B.A. Cantab.

PART I. QUADRATURE

A. Quadrature by Equi-distant Ordinates

Before we proceed to the discussion of any new formulae, it may be well to give some account of the usual methods of quadrature*.

Consider the area of any curve bounded by the ordinates at the points x_0, x_p, the curve itself and the axis of x, and let us suppose the base divided into p equal portions of length h, by the ordinates $z_0, z_1, z_2 \dots z_p$. Then for obtaining approximations to the area $\int_{x_0}^{x_p} z\,dx \left(= \int_0^{ph} z\,dx, \text{ if our origin is taken at } x_0\right)$ we may use either the extreme ordinates $z_0, z_1, z_2 \dots z_p$, or the mid-ordinates $z_{\frac{1}{2}}, z_{\frac{3}{2}}, z_{\frac{5}{2}} \dots z_{p-\frac{1}{2}}$ of the constituent trapezettes.

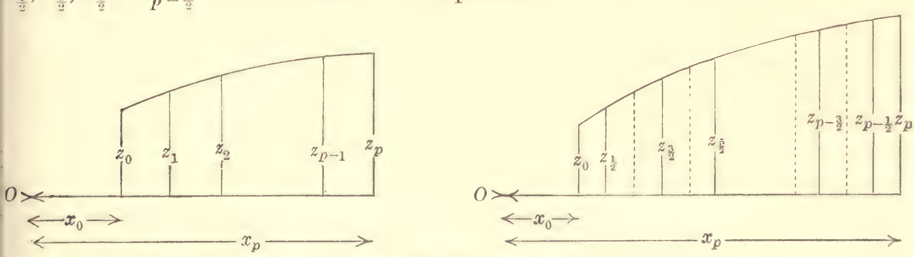


Fig. 1

The fundamental formulae are the Euler-McLaurin formulae

$$\int_{x_0}^{x_p} z\,dx = A_C - \left[\frac{1}{12}h^2\frac{dz}{dx} - \frac{h^4}{720}\frac{d^3z}{dx^3} + \frac{h^6}{30240}\frac{d^5z}{dx^5} - \dots\right]_{x_0}^{x_p} \quad \dots\dots (\alpha),$$

$$\int_{x_0}^{x_p} z\,dx = A_T + \left[\frac{1}{24}h^2\frac{dz}{dx} - \frac{7h^4}{5760}\frac{d^3z}{dx^3} + \frac{31h^6}{967680}\frac{d^5z}{dx^5} - \dots\right]_{x_0}^{x_p} \dots\dots (\beta),$$

* This collection of well-known formulae is taken from *Biometrika*, Vol. I, pp. 273—278. I am indebted to Professor Pearson for permission to use it.

where $$A_C = h\left(\tfrac{1}{2}z_0 + z_1 + z_2 + \ldots + z_{p-1} + \tfrac{1}{2}z_p\right),$$
$$A_T = h\left(z_{\frac{1}{2}} + z_{\frac{3}{2}} + z_{\frac{5}{2}} + \ldots + z_{p-\frac{3}{2}} + z_{p-\frac{1}{2}}\right),$$
and the values of the expressions in square brackets have to be calculated at the points x_0, x_p and the former subtracted from the latter.

For purposes of practical work the differential coefficients in these formulae can be replaced by differences, when we reach

$$\int_{x_0}^{x_p} z\,dx = A_C + h\left(\gamma_1\Delta - \gamma_2\Delta^2 + \gamma_3\Delta^3 - \gamma_4\Delta^4 + \ldots\right)(z_0 + z_p),$$

$$\int_{x_0}^{x_p} z\,dx = A_T - h\left(\gamma_1'\Delta - \gamma_2'\Delta^2 + \gamma_3'\Delta^3 - \gamma_4'\Delta^4 + \ldots\right)(z_{\frac{1}{2}} + z_{p-\frac{1}{2}}).$$

In these formulae Δ operating on z_p and $z_{p-\frac{1}{2}}$ must be taken backwards, i.e. $\Delta z_p = z_{p-1} - z_p$ and $\Delta z_{p-\frac{1}{2}} = z_{p-\frac{3}{2}} - z_{p-\frac{1}{2}}$ while $\Delta z_0 = z_1 - z_0$, $\Delta z_{\frac{1}{2}} = z_{\frac{3}{2}} - z_{\frac{1}{2}}$.

The values of the coefficients γ are:

$\gamma_1 = \cdot 083{,}3333$	$\gamma_1' = \cdot 041{,}6667$
$\gamma_2 = \cdot 041{,}6667$	$\gamma_2' = \cdot 041{,}6667$
$\gamma_3 = \cdot 026{,}3889$	$\gamma_3' = \cdot 038{,}7153$
$\gamma_4 = \cdot 018{,}7500$	$\gamma_4' = \cdot 035{,}7639$
$\gamma_5 = \cdot 014{,}2692$	$\gamma_5' = \cdot 033{,}1918$
$\gamma_6 = \cdot 011{,}3674$	$\gamma_6' = \cdot 030{,}9989$
$\gamma_7 = \cdot 009{,}3565$	$\gamma_7' = \cdot 029{,}1253$
$\gamma_8 = \cdot 007{,}8925$	$\gamma_8' = \cdot 027{,}5110$
$\gamma_9 = \cdot 006{,}7858$	$\gamma_9' = \cdot 026{,}1066$
$\gamma_{10} = \cdot 005{,}9241$	$\gamma_{10}' = \cdot 024{,}8732$
$\gamma_{11} = \cdot 005{,}2367$	$\gamma_{11}' = \cdot 023{,}7807$
$\gamma_{12} = \cdot 004{,}6775$	$\gamma_{12}' = \cdot 022{,}8052$
$\gamma_{13} = \cdot 004{,}2150$	
$\gamma_{14} = \cdot 003{,}8269$	

Professor Pearson has pointed out that these Euler-McLaurin formulae (the correction terms being, as they usually are, small) give equal weight to all the ordinates except the first and last, in the principal portions A_C and A_T of the formulae; a circumstance which is of great importance when the ordinates are observations which are liable to error.

Although they will give the best possible results if we go to the complete system of differences for the ordinates, to do so involves very great labour, and if only three or four γ's are used, they are not the best coefficients by which to multiply the successive differences.

For these reasons the following formulae, in which the number of ordinates used is a multiple of 2, 3, 4, 6 ... etc., are preferable.

Simpson's Rule ($2p$ elements).

$$\int_{x_0}^{x_{2p}} z\,dx = \tfrac{1}{3}h\,\{z_0 + 2(z_2 + z_4 + \ldots + z_{2p-2}) + z_{2p} + 4(z_1 + z_3 + \ldots + z_{2p-1})\} \ldots (\gamma).$$

Newton's Rule ($3p$ elements).

$$\int_{x_0}^{x_{3p}} z\,dx = \tfrac{3}{8}h\,\{z_0 + 3(z_1 + z_2 + z_4 + z_5 + z_7 + z_8 + \ldots) + z_{3p} + 2(z_3 + z_6 + \ldots + z_{3p-3})\} \ldots\ldots (\delta).$$

Boole's Rule ($4p$ elements).

$$\int_{x_0}^{x_{4p}} z\,dx = \tfrac{2}{45}h\,\{7z_0 + 14(z_4 + z_8 + \ldots + z_{4p-4}) + 7z_{4p} + 32(z_1 + z_3 + z_5 + \ldots + z_{4p-1}) + 12(z_2 + z_6 + z_{10} + \ldots + z_{4p-2})\} \ldots\ldots (\epsilon).$$

Weddle's Rule ($6p$ elements).

$$\int_{x_0}^{x_{6p}} z\,dx = \frac{3h}{10}\{z_0 + z_2 + z_4 + z_8 + z_{10} + \ldots + z_{6p-2} + z_{6p} + 2(z_6 + z_{12} + \ldots + z_{6p-6}) + 5(z_1 + z_5 + z_7 + z_{11} + \ldots + z_{6p-1}) + 6(z_3 + z_9 + z_{15} + \ldots + z_{6p-3})\} \ldots\ldots (\zeta).$$

These formulae, whose exactness is in the order in which they are written down, also have their disadvantages. It is not always possible to select beforehand the number of elements to be used, and this number may not fit in with any of the above formulae; secondly, all the ordinates are not equally weighted which is a disadvantage if they are observational quantities subject to errors. Formulae (ϵ) and (ζ), particularly (ζ), have been found to give very good results for continuous mathematical functions, but they are not so good for observational data.

For this purpose the following formulae of Dr Sheppard's are better. The corrective terms are so chosen, as to make the first term, in the Euler-McLaurin formulae, which they neglect, as small as possible*.

Case (i). *Bounding ordinates or chordal area known.*

(*a*) *One Difference:*

$$\text{Area} = A_C + \frac{1}{12}\frac{p}{p-1}\{(z_1 - z_0) - (z_p - z_{p-1})\}\,h \ldots\ldots (\eta).$$

(*b*) *Two Differences:*

$$\text{Area} = A_C + \frac{1}{120}\frac{p(15p-26)}{(p-1)(p-2)}\{(z_1 - z_0) - (z_p - z_{p-1})\}\,h - \frac{1}{120}\frac{p(5p-6)}{(p-2)(p-3)}\{(z_2 - z_1) - (z_{p-1} - z_{p-2})\}\,h \quad \ldots (\theta).$$

* See *Proceedings of the London Mathematical Society*, Vol. XXXII, p. 270.

(c) *Three Differences:*

$$\text{Area} = A_C + \frac{1}{5040}\frac{p(763p^2 - 3444p + 3636)}{(p-1)(p-2)(p-3)}\{(z_1 - z_0) - (z_p - z_{p-1})\}h$$
$$- \frac{1}{1260}\frac{p(119p^2 - 504p + 432)}{(p-2)(p-3)(p-4)}\{(z_2 - z_1) - (z_{p-1} - z_{p-2})\}h$$
$$+ \frac{1}{5040}\frac{p(113p^2 - 462p + 360)}{(p-3)(p-4)(p-5)}\{(z_3 - z_2) - (z_{p-2} - z_{p-3})\}h\,^* \quad \ldots\ldots\ldots\ldots(\iota).$$

Case (ii). *Mid-ordinates or tangential area known.*

(a) *One Difference:*

$$\text{Area} = A_T - \frac{1}{24}\frac{p}{(p-2)}\{(z_{\frac{3}{2}} - z_{\frac{1}{2}}) - (z_{p-\frac{1}{2}} - z_{p-\frac{3}{2}})\}h \quad \ldots\ldots\ldots(\kappa).$$

(b) *Two Differences:*

$$\text{Area} = A_T - \frac{1}{960}\frac{p(80p - 177)}{(p-2)(p-3)}\{(z_{\frac{3}{2}} - z_{\frac{1}{2}}) - (z_{p-\frac{1}{2}} - z_{p-\frac{3}{2}})\}h$$
$$+ \frac{1}{960}\frac{p(40p - 57)}{(p-3)(p-4)}\{(z_{\frac{5}{2}} - z_{\frac{3}{2}}) - (z_{p-\frac{3}{2}} - z_{p-\frac{5}{2}})\}h \quad \ldots(\lambda).$$

This formula, Professor Pearson has pointed out, has many advantages. It is more exact than (κ) and sufficiently so for practical purposes, though not quite as exact as (μ). Further, all the ordinates that occur in the expression A_T are equally weighted, which is not the case with A_C, where the end ordinates only have half-weight. It may be written

$$\text{Area} = A_T - P\{(z_{\frac{3}{2}} - z_{\frac{1}{2}}) - (z_{p-\frac{1}{2}} - z_{p-\frac{3}{2}})\}h + Q\{(z_{\frac{5}{2}} - z_{\frac{3}{2}}) - (z_{p-\frac{3}{2}} - z_{p-\frac{5}{2}})\}h,$$

where the following are the values of P and Q:

p	P	Q	p	P	Q
8	·128,6111	·109,5833	15	·102,4639	·064,2756
9	·121,2054	·094,6875	16	·101,0073	·062,2863
10	·115,8854	·085,0694	17	·099,7569	·060,6170
11	·111,8779	·078,3668	18	·098,6719	·059,1964
12	·108,7500	·073,4375	19	·097,7214	·057,9731
13	·106,2405	·069,6644	20	·096,8818	·056,9087
14	·104,1825	·066,6856			

This formula gives results which are, as a rule, better than Simpson's, and only those ordinates are weighted which occur in the corrective terms.

* The coefficients in this formula have been tabulated, for values of p from 6 to 100, by P. F. Everitt. See *Biometrika*, Vol. XII, p. 283.

(c) *Three Differences:*

$$\text{Area} = A_T - \frac{1}{80640}\,\frac{p(9842p^2 - 53970p + 70407)}{(p-2)(p-3)(p-4)}\{(z_{\frac{3}{2}} - z_{\frac{1}{2}}) - (z_{p-\frac{1}{2}} - z_{p-\frac{3}{2}})\}\,h$$
$$+ \frac{1}{40320}\,\frac{p(4802p^2 - 23016p + 22905)}{(p-3)(p-4)(p-5)}\{(z_{\frac{5}{2}} - z_{\frac{3}{2}}) - (z_{p-\frac{3}{2}} - z_{p-\frac{5}{2}})\}\,h$$
$$- \frac{1}{80640}\,\frac{p(3122p^2 - 12222p + 10935)}{(p-4)(p-5)(p-6)}\{(z_{\frac{7}{2}} - z_{\frac{5}{2}}) - (z_{p-\frac{5}{2}} - z_{p-\frac{7}{2}})\}\,h \quad \ldots\ldots\ldots\ldots(\mu).$$

This is rather complicated to use in practice, though with the aid of a calculating machine, the coefficients might easily be tabulated.

Case (iii). *Mid-ordinates and two extreme ordinates known.*

(a) *One Difference:*

$$\text{Area} = A_T - \frac{1}{12}\,\frac{2p}{2p-1}\{(z_{\frac{1}{2}} - z_0) - (z_p - z_{p-\frac{1}{2}})\}\,h \quad \ldots\ldots\ldots\ldots(\nu).$$

(b) *Two Differences:*

$$\text{Area} = A_T - \frac{1}{180}\,\frac{2p(40p-57)}{(2p-1)(2p-3)}\{(z_{\frac{1}{2}} - z_0) - (z_p - z_{p-\frac{1}{2}})\}\,h$$
$$+ \frac{1}{180}\,\frac{2p(5p-6)}{(2p-2)(2p-3)}\{(z_{\frac{3}{2}} - z_{\frac{1}{2}}) - (z_{p-\frac{1}{2}} - z_{p-\frac{3}{2}})\}\,h \ldots(\xi).$$

Case (iv). *Bounding ordinates, with the two mid-ordinates only, of the terminal elements.*

(a) *One Difference:*

$$\text{Area} = A_C + \frac{1}{6}\,\frac{2p}{2p-1}\{(z_{\frac{1}{2}} - z_0) - (z_p - z_{p-\frac{1}{2}})\}\,h \quad \ldots\ldots\ldots\ldots(o).$$

(b) *Two Differences:*

$$\text{Area} = A_C + \frac{1}{120}\,\frac{2p(30p-29)}{(2p-1)(p-1)}\{(z_{\frac{1}{2}} - z_0) - (z_p - z_{p-\frac{1}{2}})\}\,h$$
$$- \frac{1}{120}\,\frac{2p(10p-9)}{(2p-3)(p-1)}\{(z_1 - z_{\frac{1}{2}}) - (z_{p-\frac{1}{2}} - z_{p-1})\}\,h \ldots(\pi).$$

It has been pointed out that, for fairly large values of p this is not very divergent from

$$\text{Area} = A_C + \tfrac{1}{4}\{(z_{\frac{1}{2}} - z_0) - (z_p - z_{p-\frac{1}{2}})\}\,h - \tfrac{1}{12}\{(z_1 - z_{\frac{1}{2}}) - (z_{p-\frac{1}{2}} - z_{p-1})\}\,h \quad \ldots\ldots\ldots\ldots(\rho),$$

which may be obtained directly by a double application of Simpson's formula, and is somewhat more accurate than the latter.

The following table shows the results given by the above formulae, when used to calculate $\int_0^1 \frac{dx}{1+x} = \cdot 693,147,18$. This integral is rather a favourable one on which to test an individual formula, but it may perhaps serve to compare the accuracy obtained by different formulae. Either 12 or 13 ordinates have been used in every case.

	Divergence		Divergence
(α) with four differences	$+\cdot 000,000,25$	(κ)	$-\cdot 000,014,93$
(β) „ „ „	$-\cdot 000,000,59$	(λ)	$-\cdot 000,001,26$
(γ)	$+\cdot 000,001,48$	(μ)	$-\cdot 000,000,12$
(δ)	$+\cdot 000,003,28$	(ν)	$-\cdot 000,003,91$
(ϵ)	$+\cdot 000,000,07$	(ξ)	$-\cdot 000,000,14$
(ζ)	$+\cdot 000,000,04$	(o)	$+\cdot 000,008,12$
(η)	$+\cdot 000,014,59$	(π)	$+\cdot 000,000,22$
(θ)	$+\cdot 000,000,93$	(ρ)	$+\cdot 000,001,27$
(ι)	$+\cdot 000,000,07$		

Also $A_C = \cdot 693,580,83$. $\quad \Delta = \cdot 000,433,65$.

$A_T = \cdot 692,930,49$. $\quad \Delta = -\cdot 000,216,69$*.

B. The Gaussian Method of Quadrature

No discussion of quadrature would be complete without some reference to Gauss' method, in which the ordinates are no longer taken at equal intervals, but at such intervals as make the approximation obtained as close as possible.

Suppose it be required to approximate to $\int_{-1}^{1} f(x)\,dx$. Let $P_n(x)$ be Legendre's polynomial of the nth order. Then

$$P_n(x) = \frac{1}{2^n \lfloor n} \frac{d^n}{dx^n}(x^2-1)^n$$

and $$\int_{-1}^{1} x^r P_n(x)\,dx = 0, \text{ for } r = 0, 1, 2 \ldots n-1\dagger.$$

Now suppose $f(x)$ to be a polynomial of degree not greater than $2n-1$. Then by division we have $f(x) = P_n(x)\,Q(x) + R(x)$, where $Q(x)$, $R(x)$ are of degree not greater than $n-1$.

$$\therefore \int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} P_n(x)\,Q(x)\,dx + \int_{-1}^{1} R(x)\,dx;$$

but $$\int_{-1}^{1} P_n(x)\,Q(x)\,dx = 0,$$

$$\therefore \int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} R(x)\,dx.$$

* See Pearson, 'On the systematic fitting of curves to observations and measurements,' *Biometrika*, Vol. I, p. 265.

† See Whittaker and Watson's *Modern Analysis*, pp. 296—300.

Now $f(x) = R(x)$ for the values $x = a_1, a_2 \dots a_n$ which are the roots of $P_n(x) = 0$.

But by Lagrange's interpolation formula*

$$R(x) = \sum_{r=1}^{r=n} A_r R(a_r),$$

where
$$A_r = \frac{P_n(x)}{P'_n(a_r)(x - a_r)},$$

whence we deduce the quadrature formula

$$\int_{-1}^{1} f(x)\,dx = \sum_{r=1}^{r=n} B_r f(a_r),$$

where
$$B_r = \int_{-1}^{1} \frac{P_n(x)\,dx}{P'_n(a_r)(x - a_r)},$$

or we have a quadrature formula involving n ordinates only, which is exact for a function of degree not greater than $2n - 1$, whereas, if we had taken the ordinates at equal intervals we should have required $2n$ to obtain the integral exactly.

Gauss actually considered the integral $\int_0^1 f(x)\,dx$, which only requires a slight modification of the above argument, and his result for the case of seven ordinates is as follows: $\int_0^1 f(x)\,dx = \sum_{r=1}^{r=7} R_r f(a_r)$, where we have the following values for the R's and a's†.

r	a_r	R_r
1	·02544 6043829	·06474 2483084
2	·12923 4407200	·13985 2695745
3	·29707 7424311	·19091 5025253
4	·5	·20897 9519837
5	·70292 2575689	·19091 5025253
6	·87076 5592800	·13985 2695745
7	·97455 3956171	·06474 2483084

For the integral $\int_a^b f(x)\,dx$ the formula becomes

$$\int_a^b f(x)\,dx = \sum_{r=1}^{r=7} (b-a) f\{a + (b-a)\,a_r\} R_r,$$

a_r and R_r having the above values.

* See *Tracts for Computers*, II, p. 17.

† Gauss, *Werke*, III, p. 195, Göttingen, 1866. Gauss gives these values to 15 decimals, and we have here given them to 12. Naturally it will not usually be necessary to use all 12.

We have tried this formula on several mathematical functions.

(i) $\int_0^1 \frac{1}{1+x}\,dx = \cdot693{,}147{,}18$ (correct to 8 figures).

(ii) $\int_0^3 \frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}x^2}\,dx = \cdot498{,}650{,}1$ (correct to 7 figures).

(iii) $\int_0^{\frac{1}{2}} x^{1\cdot5}(1-x)^{\cdot5}\,dx = \cdot056{,}508{,}10$ (correct to 8 figures).

(iv) $$I_{\cdot6}(16\cdot1,\ 5\cdot2) = \frac{\int_0^{\cdot6} x^{15\cdot1}(1-x)^{4\cdot2}\,dx}{\int_0^1 x^{15\cdot1}(1-x)^{4\cdot2}\,dx}.$$

This is the ratio of the incomplete Beta-function $B_{\cdot6}(16\cdot1, 5\cdot2)$ to the complete Beta-function $B(16\cdot1,\ 5\cdot2)$.

The latter function $= \frac{\Gamma(21\cdot3)}{\Gamma(16\cdot1)\,\Gamma(5\cdot2)} = 5\cdot0320065$ (by use of the tables of the Γ-function); while the whole function $I_{\cdot6}(16\cdot1,\ 5\cdot2)$ has been calculated to twelve figures, by Mr H. E. Soper*, by integration by parts and its value correct to seven figures $= \cdot0567099$.

(v) $\int_{60^\circ}^{180^\circ} (1+\cos x)\,dx = 1\cdot2283697$ (correct to seven decimals).

Using the Gauss method with seven ordinates we find:

(i) $\int_0^1 \frac{dx}{1+x} = \cdot69314719.$

This is only in error by 1 in the eighth place, a better result than is given by any of the formulae in Section A, with 12 or 13 ordinates.

(ii) $\int_0^3 \frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}x^2}\,dx = \cdot4986498$ (divergence $+\cdot0000003$).

With the same number of ordinates:

Simpson's formula gives: $\cdot4986237$ (divergence $-\cdot0000264$)
and Weddle's „ „ : $\cdot4986614$ (divergence $+\cdot0000113$),
so that Gauss' formula gives a markedly better result†.

(iii) $\int_0^{\frac{1}{2}} x^{1\cdot5}(1-x)^{\cdot5}\,dx = \cdot05650984$ (divergence $+\cdot00000174$).

Weddle's gives $\cdot05652562$ (divergence $+\cdot00001752$).

* *Tracts for Computers*, VII, p. 17.

† For comparative purposes we are only using the same number of ordinates for Weddle as for Gauss, but in evaluating such integrals we should in practice use far more frequent ordinates, and get a result better than the Gaussian, where at present we can only use seven ordinates.

(iv) $I_{.6}(16{\cdot}1, 5{\cdot}2)^* = {\cdot}0567206$ (divergence $+ {\cdot}0000107$).
Weddle's gives ·0561927 (divergence $-{\cdot}0005172$).

(v) $\int_{60}^{180^\circ} (1 + \cos x)\, dx = 1{\cdot}2283696$ (divergence $- {\cdot}0000001$).
Weddle's gives 1·2283717 (divergence $+ {\cdot}0000020$).

The results of (iv) and perhaps (iii) may be considered somewhat disappointing; but, unless we take a very large number of ordinates, no quadrature formulae will give very accurate results for functions with such rapidly changing differences, and this difficulty will generally be encountered when the function is incapable of expansion in ascending integral powers of x. The reader need only attempt to draw the curve $y = \dfrac{x^{15{\cdot}1}(1-x)^{4{\cdot}2}}{B(16{\cdot}1, 5{\cdot}2)}$ to find the explanation of this fact in the very rapid changes the function undergoes as x varies.

We are, however, of opinion that Gauss' method, if used with 12 or 13 ordinates, would give very satisfactory results, even for integrals of this type. Its use however would involve a knowledge of the roots of $P_{12}(x)$ or $P_{13}(x)$ to 10 or 12 decimal places, and we are not aware that these are available†.

We may conclude that Gauss' method is, on the whole, very satisfactory for mathematical functions that are continuous and differentiable. The drawback of the method is that its application requires more labour than equi-ordinate methods, and cannot be applied at all to observational data, since the observations would almost certainly be unobtainable at the intervals required by the theory. Further, in using a formula like Weddle's we should naturally use 13, 25 or more ordinates, but at present this is impossible by the Gaussian method.

C. Quadrature Formulae derived from Central Difference Interpolation Formulae

We will now investigate what quadrature formulae will arise from the integration of central difference interpolation formulae. We shall have two main types of formulae:

(1) Those which depend on extreme ordinates.

(2) Those which depend on mid-ordinates.

These really correspond to the mid-panel and mid-point central difference interpolation formulae‡.

* Gauss' method was here used for the numerator only, the denominator was obtained from tables of the Γ-function.

† I propose at a later date to undertake this investigation which naturally involves very considerable labour.

‡ *Tracts for Computers*, II, pp. 11—22.

(1) Extreme Ordinate Central Difference Quadrature Formulae.

We start from Everett's well-known Central Difference Formula:

$$\begin{aligned} z_\theta = {} & (1-\theta) z_s + \theta z_{s+1} - \tfrac{1}{6}\theta(1-\theta)\{(2-\theta)\delta^2 z_s + (1+\theta)\delta^2 z_{s+1}\} \\ & + \tfrac{1}{120}\theta(1-\theta)(1+\theta)(2-\theta)\{(3-\theta)\delta^4 z_s + (2+\theta)\delta^4 z_{s+1}\} \\ & - \tfrac{1}{5040}\theta(1-\theta)(1+\theta)(2-\theta)(2+\theta)(3-\theta)\{(4-\theta)\delta^6 z_s + (3+\theta)\delta^6 z_{s+1}\} \\ & + \ldots\ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}. \end{aligned}$$

Here $$\delta^2 z_s = z_{s+1} - 2z_s + z_{s-1},$$

$$\delta^{2n} z_s = \delta^{2n-2} z_{s+1} - 2\delta^{2n-2} z_{s+1} + \delta^{2n-2} z_{s-1},$$

or if E is the ordinary finite difference operator such that $EU_x = U_{x+1}$, we may write symbolically $\delta = \sqrt{E} - 1/\sqrt{E}$.

Assuming h to be the distance between the ordinates, we have

$$\begin{aligned} \int_{sh}^{(s+1)h} z\,dx &= h\int_0^1 z_\theta\, d\theta \\ &= h\left[\tfrac{1}{2}(z_s + z_{s+1}) - \tfrac{1}{24}(\delta^2 z_s + \delta^2 z_{s+1}) + \tfrac{11}{1440}(\delta^4 z_s + \delta^4 z_{s+1}) - \tfrac{191}{120960}(\delta^6 z_s + \delta^6 z_{s+1})\right] \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ldots\ldots\ldots\ldots\text{(ii)}, \end{aligned}$$

where we neglect eighth and higher order differences.

If, now, we wish to approximate to $\int_0^{mh} z\,dx$, given the $m+1$ ordinates $z_0, z_1 \ldots z_m$, we need only combine formulae such as (ii) for all the m elements of area. This would, however, be open to the objection that we should introduce, at the ends, ordinates outside the range of integration, which is undesirable as, in some cases, they might not be known or obtainable.

To get over this difficulty we have only to integrate the special first, second and third panel forms of Everett's formula*. We do so in order.

First Panel Central Difference Formula:

This is

$$\begin{aligned} z_\theta = {} & (1-\theta) z_0 + \theta z_1 - \tfrac{1}{6}\theta(1-\theta)\{(5-\theta)\delta^2 z_1 - (2-\theta)\delta^2 z_2\} \\ & - \tfrac{1}{120}\theta(1-\theta)(2-\theta)(3-\theta)\{(9-\theta)\delta^4 z_2 - (4-\theta)\delta^4 z_3\} \\ & - \tfrac{1}{5040}\theta(1-\theta)(2-\theta)(3-\theta)(4-\theta)(5-\theta)\{(13-\theta)\delta^6 z_3 - (6-\theta)\delta^6 z_4\} \\ & + \ldots\ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}. \end{aligned}$$

We obtain

$$\begin{aligned} \int_0^h z\,dx = h\Big[& \tfrac{1}{2}(z_0 + z_1) - \tfrac{1}{24}\{3\delta^2 z_1 - \delta^2 z_2\} - \tfrac{1}{1440}\{65\delta^4 z_2 - 27\delta^4 z_3\} \\ & - \tfrac{1}{120960}\{3101\delta^6 z_3 - 1375\delta^6 z_4\}\Big]\ldots\text{(iv)}. \end{aligned}$$

* These are provided in *Tracts for Computers*, II, pp. 13, 14.

Second Panel Central Difference Formula:

This is

$$z_\theta = (1-\theta) z_1 + \theta z_2 - \tfrac{1}{6}\theta(1-\theta)\{(2-\theta)\delta^2 z_1 + (1+\theta)\delta^2 z_2\}$$
$$+ \tfrac{1}{120}\theta(1-\theta)(1+\theta)(2-\theta)\{(8-\theta)\delta^4 z_2 - (3-\theta)\delta^4 z_3\}$$
$$+ \tfrac{1}{5040}\theta(1-\theta)(1+\theta)(2-\theta)(3-\theta)(4-\theta)\{(12-\theta)\delta^6 z_3 - (5-\theta)\delta^6 z_4\}$$
$$+ \ldots\ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(v)};$$

whence

$$\int_h^{2h} z\,dx = h\left[\tfrac{1}{2}(z_1+z_2) - \tfrac{1}{24}(\delta^2 z_1 + \delta^2 z_2) + \tfrac{11}{1440}\{3\delta^4 z_2 - \delta^4 z_3\}\right.$$
$$\left. + \tfrac{1}{120960}\{893\delta^6 z_3 - 351\delta^6 z_4\}\right]\ldots\text{(vi)}.$$

Third Panel Central Difference Formula:

This is

$$z_\theta = (1-\theta) z_2 + \theta z_3 - \tfrac{1}{6}\theta(1-\theta)\{(2-\theta)\delta^2 z_2 + (1+\theta)\delta^2 z_3\}$$
$$+ \tfrac{1}{120}\theta(1-\theta)(1+\theta)(2-\theta)\{(3-\theta)\delta^4 z_2 + (2+\theta)\delta^4 z_3\}$$
$$- \tfrac{1}{5040}\theta(1-\theta)(1+\theta)(2-\theta)(2+\theta)(3-\theta)\{(11-\theta)\delta^6 z_3 - (4-\theta)\delta^6 z_4\}$$
$$+ \ldots\ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(vii)};$$

whence

$$\int_{2h}^{3h} z\,dx = h\left[\tfrac{1}{2}(z_2+z_3) - \tfrac{1}{24}(\delta^2 z_2 + \delta^2 z_3) + \tfrac{11}{1440}(\delta^4 z_2 + \delta^4 z_3)\right.$$
$$\left. - \tfrac{191}{120960}\{3\delta^6 z_3 - \delta^6 z_4\}\right]\ldots\text{(viii)}.$$

We are now in a position, by combining appropriately (ii), (iv), (vi) and (viii), to obtain a formula for $\int_0^{mh} z\,dx$.

We have

$$\int_0^{mh} z\,dx = h\left[\tfrac{1}{2}z_0 + z_1 + z_2 + \ldots + z_{m-1} + \tfrac{1}{2}z_m\right.$$
$$- \tfrac{1}{24}(4\delta^2 z_1 + \delta^2 z_2 + 2\delta^2 z_3 + 2\delta^2 z_4 + \ldots + 2\delta^2 z_{m-3} + \delta^2 z_{m-2} + 4\delta^2 z_{m-1}$$
$$+ \tfrac{1}{1440}(-21\delta^4 z_2 + 38\delta^4 z_3 + 22\delta^4 z_4 + 22\delta^4 z_5 + \ldots$$
$$+ 22\delta^4 z_{m-4} + 38\delta^4 z_{m-3} - 21\delta^4 z_{m-2}$$
$$- \tfrac{1}{120960}(2972\delta^6 z_3 - 833\delta^6 z_4 + 382\delta^6 z_5 + 382\delta^6 z_6 + \ldots$$
$$\left. + 382\delta^6 z_{m-5} - 833\delta^6 z_{m-4} + 2972\delta^6 z_{m-3})\right]$$
$$= h\left[\tfrac{1}{2}z_0 + z_1 + z_2 + \ldots + z_{m-1} + \tfrac{1}{2}z_m\right.$$
$$- \tfrac{1}{24}\{2\delta^2 z_1 - \delta^2 z_2 + 2(z_0 - z_1 - z_{m-1} + z_m) - \delta^2 z_{m-2} + 2\delta^2 z_{m-1}\}$$
$$+ \tfrac{1}{1440}\{-43\delta^4 z_2 + 16\delta^4 z_3 + 22(\delta^2 z_1 - \delta^2 z_2 - \delta^2 z_{m-2} + \delta^2 z_{m-1})$$
$$+ 16\delta^4 z_{m-3} - 43\delta^4 z_{m-2}\}$$
$$- \tfrac{1}{120960}\{2590\delta^6 z_3 - 1215\delta^6 z_4 + 382(\delta^4 z_2 - \delta^4 z_3 - \delta^4 z_{m-3} + \delta^4 z_{m-2})$$
$$\left. - 1215\delta^6 z_{m-4} + 2590\delta^6 z_{m-3}\}\right]\ldots\text{(ix)},$$

which is correct up to and including seventh order differences.

This may also be put in the form

$$
\begin{aligned}
h\Big[&\tfrac{1}{2}z_0 + z_1 + z_2 + \ldots + z_{m-1} + \tfrac{1}{2}z_m \\
&- \tfrac{1}{24}\{4(z_0+z_m) - 7(z_1+z_{m-1}) + 4(z_2+z_{m-2}) - (z_3+z_{m-3})\} \\
&+ \tfrac{1}{1440}\{-21(\delta^2 z_1 + \delta^2 z_{m-1}) + 80(\delta^2 z_2 + \delta^2 z_{m-2}) - 75(\delta^2 z_3 + \delta^2 z_{m-3}) \\
&\qquad + 16(\delta^2 z_4 + \delta^2 z_{m-4})\} \\
&- \tfrac{1}{120960}\{2972(\delta^4 z_2 + \delta^4 z_{m-2}) - 6777(\delta^4 z_3 + \delta^4 z_{m-3}) + 5020(\delta^4 z_4 + \delta^4 z_{m-4}) \\
&\qquad - 1215(\delta^4 z_5 + \delta^4 z_{m-5})\}\Big] \ldots \text{(ix) } bis,
\end{aligned}
$$

which although it contains no higher differences than the 4th is correct up to and including 7th order differences.

By expressing these results in terms of the ordinates, stopping at the 1st, 2nd, 3rd, 4th lines of expression (ix), we may obtain quadrature formulae, containing ordinates only, respectively correct up to and including 1st, 3rd, 5th and 7th order differences.

These are:

$$
\left\{
\begin{aligned}
&(\alpha)\quad h\left(\tfrac{1}{2}z_0 + z_1 + z_2 + \ldots\ldots + z_{m-1} + \tfrac{1}{2}z_m\right), \\
&(\beta)\quad \frac{h}{24}[8z_0 + 31z_1 + 20z_2 + 25z_3 + 24(z_4 + \ldots + z_{m-4}) + 25z_{m-3} \\
&\qquad\qquad + 20z_{m-2} + 31z_{m-1} + 8z_m], \\
&(\gamma)\quad \frac{h}{1440}[459z_0 + 1982z_1 + 944z_2 + 1746z_3 + 1333z_4 \\
&\qquad\qquad + 1456z_5 + 1440(z_6 + \ldots + z_{m-6}) + 1456z_{m-5} \\
&\qquad\qquad + 1333z_{m-4} + 1746z_{m-3} + 944z_{m-2} + 1982z_{m-1} + 459z_m], \\
&(\delta)\quad \frac{h}{120960}[35584(z_0+z_m) + 185153(z_1+z_{m-1}) + 29336(z_2+z_{m-2}) \\
&\qquad\qquad + 220509(z_3+z_{m-3}) + 46912(z_4+z_{m-4}) + 156451(z_5+z_{m-5}) \\
&\qquad\qquad + 111080(z_6+z_{m-6}) + 122175(z_7+z_{m-7}) + 120960(z_8 + \ldots + z_{m-8})]
\end{aligned}
\right.
$$

$$\ldots\ldots\ldots\ldots\text{(x)}.$$

These formulae are perhaps more convenient than the difference formulae when a calculating machine is available, though it must be remembered that it is always necessary to examine the differences, to know to what order of differences it is necessary to proceed in the approximation. To apply formulae (β), (γ), (δ) we require at least 8, 12 and 16 ordinates respectively.

We have applied these formulae to $\int_0^1 \frac{dx}{1+x}$ to compare their accuracy with the formulae previously given. 13 ordinates were used, except in the case of (x) δ (which is equivalent to (ix) retaining all the differences) when we used 17.

The results are:

(x) (α) or the chordal area $= \cdot693{,}580{,}83$ (divergence $+\ \cdot00043365$),
(x) (β) or (ix) up to terms of $\delta^2 = \cdot693{,}145{,}94$ (divergence $-\ \cdot00000124$),
(x) (γ) or (ix) up to terms in $\delta^4 = \cdot693{,}147{,}30$ (divergence $+\ \cdot00000012$),
(x) (δ) or (ix) up to terms in $\delta^6 = \cdot693{,}147{,}18(1)$ (divergence 0 to 9 figures).

The correct value $= \log_e 2 = 693{,}147{,}180{,}6$ to ten figures. (x) (δ) is the most accurate result obtained by any quadrature formula here given; as an example we show the working in full.

x	$z=1/1+x$	δ^2	δ^4	δ^6
0	1·0			
·0625	·94117647	·00653595		
·1250	·88888889	·00550395	·00020642	
·1875	·84210526	·00467837	·00015723	·00001350
·2500	·80000000	·00401002	·00012154	·00000922
·3125	·76190476	·00346321	·00009507	·00000673
·3750	·72727273	·00301147	·00007533	·00000459
·4375	·69565217	·00263506	·00006018	·00000366
·5000	·66666667	·00231883	·00004869	·00000240
·5625	·64000000	·00205129	·00003960	·00000209
·6250	·61538462	·00182335	·00003260	·00000132
·6875	·59259259	·00162801	·00002692	·00000121
·7500	·57142857	·00145959	·00002245	·00000088
·8125	·55172414	·00131362	·00001886	·00000060
·8750	·53333333	·00118651	·00001587	
·9375	·51612903	·00107527		
1·0000	·50000000			

Not all the differences are required in the calculation, but an examination of all the 6th differences is, to some extent, a check on the work.

We have

$$\tfrac{1}{2}z_0+z_1+z_2+\ldots\ldots+z_{m-1}+\tfrac{1}{2}z_m = 11\cdot09425923$$

$$\left.\begin{array}{r} 2(\delta^2z_1+\delta^2z_{m-1}) = \cdot01522244\\ -(\delta^2z_2+\delta^2z_{m-2}) = -\cdot00669046\\ 2(z_0-z_1-z_{m-1}+z_m) = \cdot08538900\end{array}\right\}\cdot09392098\,(\div -24) = -\cdot00391337$$

$$\left.\begin{array}{r} -43(\delta^4z_2+\delta^4z_{m-2}) = -\cdot00955847\\ 16(\delta^4z_3+\delta^4z_{m-3}) = \cdot00281744\\ 22(\delta^2z_1-\delta^2z_2-\delta^2z_{m-2}+\delta^2z_{m-1}) = \cdot02025672\end{array}\right\}\cdot01351569\,(\div 1440) = \cdot00000939$$

$$\left.\begin{array}{r} 2590(\delta^6z_3+\delta^6z_{m-3}) = \cdot03651900\\ -1215(\delta^6z_4+\delta^6z_{m-4}) = -\cdot01227150\\ 382(\delta^4z_2-\delta^4z_3-\delta^4z_{m-3}+\delta^4z_{m-2}) = \cdot01764840\end{array}\right\}\cdot04189590\,(\div -120960) = -\cdot00000035$$

$$11\cdot09035490$$

$$h = \cdot0625$$

$$\therefore \int_0^1 \frac{dx}{1+x} = \cdot693147181$$

If the computer has a calculating machine at his disposal, he can use (x)(δ) and multiply the ordinates by their appropriate coefficients in one continuous operation on the machine. The actual process is probably slightly quicker than the above.

As a more stringent test we may consider the integral

$$I_{\cdot 6}(16{\cdot}1,\ 5{\cdot}2) = \int_0^{\cdot 6} \frac{x^{15\cdot 1}(1-x)^{4\cdot 2}\,dx}{B(16{\cdot}1,\ 5{\cdot}2)}.$$

Using 9-figure logarithms we obtain the following ordinates:

x	$x^{15\cdot 1}(1-x)^{4\cdot 2}/B(16{\cdot}1, 5{\cdot}2)$	x	$x^{15\cdot 1}(1-x)^{4\cdot 2}/B(16{\cdot}1, 5{\cdot}2)$
0	0	·3375	·0014385
·0375	0	·3750	·0055281
·0750	0	·4125	·0179780
·1125	0	·4500	·0507036
·1500	0	·4875	·1262227
·1875	·0000005	·5250	·2808828
·2250	·0000061	·5625	·5635916
·2625	·0000508	·6000	1·0250437
·3000	·0003062		

Whence using formula (x)(δ) we find

$$I_{\cdot 6}(16{\cdot}1,\ 5{\cdot}2) = {\cdot}0567101.$$

The correct answer $= {\cdot}0567099,$

or our divergence is ·0000002, a good result considering the nature of the function. A formula which gives a satisfactory result for such an integral will almost certainly be adequate for most ordinary cases that are likely to arise.

(2) Mid-ordinate Central Difference Quadrature Formulae.

The simplest method of procedure, in order to obtain these formulae, is to integrate the interpolation formulae (i), (iii), (v) and (vii) between $-\frac{1}{2}$ and $\frac{1}{2}$ for θ, then to change the origin to the point $(-\frac{1}{2}h, 0)$ and finally to combine the results to obtain $\int_0^{mh} z\,dx$ in terms of mid-ordinates.

Formula (i).

$$\begin{aligned} z_\theta = (1-\theta)\,z_s + \theta z_{s+1} &- \tfrac{1}{6}\theta(1-\theta)\{(2-\theta)\,\delta^2 z_s + (1+\theta)\,\delta^2 z_{s+1}\} \\ &+ \tfrac{1}{120}\theta(1-\theta)(1+\theta)(2-\theta)\{(3-\theta)\,\delta^4 z_s + (2+\theta)\,\delta^4 z_{s+1}\} \\ &- \tfrac{1}{5040}\theta(1-\theta)(1+\theta)(2-\theta)(2+\theta)(3-\theta)\{(4-\theta)\,\delta^6 z_s + (3+\theta)\,\delta^6 z_{s+1}\} \\ &+ \ldots\ldots\end{aligned}$$

We have
$$\int_{(s-\frac{1}{2})h}^{(s+\frac{1}{2})h} z\,dx = h\int_{-\frac{1}{2}}^{\frac{1}{2}} z_\theta\,d\theta.$$

$\therefore$ Remembering $\int_{-\frac{1}{2}}^{\frac{1}{2}} \theta^{2r-1}d\theta = 0$ for all integral values of r,

$$\int_{(s-\frac{1}{2})h}^{(s+\frac{1}{2})h} z\,dx = h\left[z_s + \tfrac{1}{24}\delta^2 z_s - \tfrac{17}{5760}\delta^4 z_s + \tfrac{367}{967680}\delta^6 z_s\right] \quad \text{.........(xi)}$$

(neglecting eighth and higher order differences).

Formula (iii).

$$\begin{aligned} z_\theta = (1-\theta)\,z_0 + \theta z_1 &- \tfrac{1}{6}\theta(1-\theta)\{(5-\theta)\,\delta^2 z_1 - (2-\theta)\,\delta^2 z_2\} \\ &- \tfrac{1}{120}\theta(1-\theta)(2-\theta)(3-\theta)\{(9-\theta)\,\delta^4 z_2 - (4-\theta)\,\delta^4 z_3\} \\ &- \tfrac{1}{5040}\theta(1-\theta)(2-\theta)(3-\theta)(4-\theta)(5-\theta)\{(13-\theta)\,\delta^6 z_3 - (6-\theta)\,\delta^6 z_4\} \\ &+ \ldots\ldots. \end{aligned}$$

We obtain

$$\begin{aligned} \int_{-\frac{1}{2}h}^{\frac{1}{2}h} z\,dx = h\,[z_0 &+ \tfrac{1}{24}(2\delta^2 z_1 - \delta^2 z_2) + \tfrac{1}{5760}(429\delta^4 z_2 - 206\delta^4 z_3) \\ &+ \tfrac{1}{967680}(62116\delta^6 z_3 - 29997\delta^6 z_4)] \ldots\text{(xii)} \end{aligned}$$

(neglecting eighth and higher order differences).

Formula (v).

$$\begin{aligned} z_\theta = (1-\theta)\,z_1 + \theta z_2 &- \tfrac{1}{6}\theta(1-\theta)\{(2-\theta)\,\delta^2 z_1 + (1+\theta)\,\delta^2 z_2\} \\ &+ \tfrac{1}{120}\theta(1-\theta)(1+\theta)(2-\theta)\{(8-\theta)\,\delta^4 z_2 - (3-\theta)\,\delta^4 z_3\} \\ &+ \tfrac{1}{5040}\theta(1-\theta)(1+\theta)(2-\theta)(3-\theta)(4-\theta)\{(12-\theta)\,\delta^6 z_3 - (5-\theta)\,\delta^6 z_4\} \\ &+ \ldots\ldots. \end{aligned}$$

Whence

$$\begin{aligned} \int_{\frac{1}{2}h}^{\frac{3}{2}h} z\,dx = h\,[z_1 &+ \tfrac{1}{24}\delta^2 z_1 - \tfrac{17}{5760}(2\delta^4 z_2 - \delta^4 z_3) \\ &- \tfrac{1}{967680}\{4611\delta^6 z_3 - 2122\delta^6 z_4\}] \ldots\text{(xiii)} \end{aligned}$$

(neglecting eighth and higher order differences).

Formula (vii).

$$\begin{aligned} z_\theta = (1-\theta)\,z_2 + \theta z_3 &- \tfrac{1}{6}\theta(1-\theta)\{(2-\theta)\,\delta^2 z_2 + (1+\theta)\,\delta^2 z_3\} \\ &+ \tfrac{1}{120}\theta(1-\theta)(1+\theta)(2-\theta)\{(3-\theta)\,\delta^4 z_2 + (2+\theta)\,\delta^4 z_3\} \\ &- \tfrac{1}{5040}\theta(1-\theta)(1+\theta)(2-\theta)(2+\theta)(3-\theta)\{(11-\theta)\,\delta^6 z_3 - (4-\theta)\,\delta^6 z_4\} \\ &+ \ldots\ldots. \end{aligned}$$

Whence

$$\int_{\frac{3}{2}h}^{\frac{5}{2}h} z\,dx = h\left[z_2 + \tfrac{1}{24}\delta^2 z_2 - \tfrac{17}{5760}\delta^4 z_2 + \tfrac{367}{967680}\{2\delta^6 z_3 - \delta^6 z_4\}\right] \ldots\text{(xiv)}.$$

If we now change the origin to the point $-\frac{1}{2}h$, 0 and appropriately combine (xi), (xii), (xiii) and (xiv) (with the changed origin) for each of the m elements of area we have:

$$\begin{aligned}
\int_0^{mh} z\,dx = h\,[&z_{\frac{1}{2}} + z_{\frac{3}{2}} + z_{\frac{5}{2}} + \ldots + z_{m-\frac{1}{2}} \\
&+ \tfrac{1}{24}\{3\delta^2 z_{\frac{3}{2}} + \delta^2 z_{\frac{7}{2}} + \delta^2 z_{\frac{9}{2}} + \ldots + \delta^2 z_{m-\frac{7}{2}} + 3\delta^2 z_{m-\frac{3}{2}}\} \\
&+ \tfrac{1}{5760}\{378\delta^4 z_{\frac{5}{2}} - 206\delta^4 z_{\frac{7}{2}} - 17(\delta^4 z_{\frac{9}{2}} + \ldots + \delta^4 z_{m-\frac{9}{2}}) \\
&\qquad - 206\delta^4 z_{m-\frac{7}{2}} + 378\delta^4 z_{m-\frac{5}{2}}\} \\
&+ \tfrac{1}{967680}\{58606(\delta^6 z_{\frac{7}{2}} + \delta^6 z_{m-\frac{7}{2}}) - 27875(\delta^6 z_{\frac{9}{2}} + \delta^6 z_{m-\frac{9}{2}}) \\
&\qquad + 367(\delta^6 z_{\frac{11}{2}} + \delta^6 z_{\frac{13}{2}} + \ldots + \delta^6 z_{m-\frac{11}{2}})\}] \\
&+ \ldots\ldots \\
= h\,[&z_{\frac{1}{2}} + z_{\frac{3}{2}} + z_{\frac{5}{2}} + \ldots + z_{m-\frac{1}{2}} \\
&+ \tfrac{1}{24}\{2(\delta^2 z_{\frac{3}{2}} + \delta^2 z_{m-\frac{3}{2}}) - (\delta^2 z_{\frac{5}{2}} + \delta^2 z_{m-\frac{5}{2}}) + (z_{\frac{1}{2}} - z_{\frac{3}{2}} - z_{m-\frac{3}{2}} + z_{m-\frac{1}{2}})\} \\
&+ \tfrac{1}{5760}\{395(\delta^4 z_{\frac{5}{2}} + \delta^4 z_{m-\frac{5}{2}}) - 189(\delta^4 z_{\frac{7}{2}} + \delta^4 z_{m-\frac{7}{2}}) \\
&\qquad - 17(\delta^2 z_{\frac{3}{2}} - \delta^2 z_{\frac{5}{2}} - \delta^2 z_{m-\frac{5}{2}} + \delta^2 z_{m-\frac{3}{2}})\} \\
&+ \tfrac{1}{967680}\{58239(\delta^6 z_{\frac{7}{2}} + \delta^6 z_{m-\frac{7}{2}}) - 28242(\delta^6 z_{\frac{9}{2}} + \delta^6 z_{m-\frac{9}{2}}) \\
&\qquad + 367(\delta^4 z_{\frac{5}{2}} - \delta^4 z_{\frac{7}{2}} - \delta^4 z_{m-\frac{7}{2}} + \delta^4 z_{m-\frac{5}{2}})\}] \\
&+ \ldots\ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots \text{(xv)},
\end{aligned}$$

which is correct up to and including 7th order differences.

By expressing this result in terms of ordinates only we may obtain four formulae according as we stop after including 1st, 3rd, 5th and 7th order differences:

$$\left\{\begin{aligned}
&(\alpha)\quad h(z_{\frac{1}{2}} + z_{\frac{3}{2}} + z_{\frac{5}{2}} + \ldots + z_{m-\frac{1}{2}}) \text{ the tangential area,} \\
&(\beta)\quad \frac{h}{24}[27(z_{\frac{1}{2}} + z_{m-\frac{1}{2}}) + 18(z_{\frac{3}{2}} + z_{m-\frac{3}{2}}) + 28(z_{\frac{5}{2}} + z_{m-\frac{5}{2}}) \\
&\qquad\qquad + 23(z_{\frac{7}{2}} + z_{m-\frac{7}{2}}) + 24(z_{\frac{9}{2}} + z_{\frac{11}{2}} + \ldots + z_{m-\frac{9}{2}})], \\
&(\gamma)\quad \frac{h}{5760}[6858(z_{\frac{1}{2}} + z_{m-\frac{1}{2}}) + 2602(z_{\frac{3}{2}} + z_{m-\frac{3}{2}}) + 9795(z_{\frac{5}{2}} + z_{m-\frac{5}{2}}) \\
&\qquad\qquad + 2823(z_{\frac{7}{2}} + z_{m-\frac{7}{2}}) + 6911(z_{\frac{9}{2}} + z_{m-\frac{9}{2}}) + 5571(z_{\frac{11}{2}} + z_{m-\frac{11}{2}}) \\
&\qquad\qquad + 5760(z_{\frac{13}{2}} + z_{\frac{15}{2}} + \ldots + z_{m-\frac{13}{2}})], \\
&(\delta)\quad \frac{h}{967680}[1210750(z_{\frac{1}{2}} + z_{m-\frac{1}{2}}) + 57625(z_{\frac{3}{2}} + z_{m-\frac{3}{2}}) \\
&\qquad\qquad + 2692267(z_{\frac{5}{2}} + z_{m-\frac{5}{2}}) - 1117816(z_{\frac{7}{2}} + z_{m-\frac{7}{2}}) \\
&\qquad\qquad + 2601308(z_{\frac{9}{2}} + z_{m-\frac{9}{2}}) + 162497(z_{\frac{11}{2}} + z_{m-\frac{11}{2}}) \\
&\qquad\qquad + 1195371(z_{\frac{13}{2}} + z_{m-\frac{13}{2}}) + 939438(z_{\frac{15}{2}} + z_{m-\frac{15}{2}}) \\
&\qquad\qquad + 967680(z_{\frac{17}{2}} + z_{\frac{19}{2}} + \ldots + z_{m-\frac{17}{2}})] \ldots\ldots\ldots \text{(xvi)}.
\end{aligned}\right.$$

It is worth noting that the 4th coefficient from either end in (δ) has a negative sign.

Applied to $\int_0^1 \frac{dx}{1+x}$ these formulae give the following results:

(xvi) (α) or the tangential area ·692,930,50 (divergence − ·00021668),

(xvi) (β) or (xv) up to 3rd differences ·693,145,51 (divergence − ·00000167),

(xvi) (γ) or (xv) up to 5th differences ·693,147,00 (divergence − ·00000018),

(xvi) (δ) or (xv) up to 7th differences ·693,147,17 (divergence − ·00000001).

12 ordinates were used for (α), (β), (γ) and 16 for (δ).

We note (xvi) (α) is rather better than ... ... (x) (α),

(xvi) (β) is not quite so good as ... ... (x) (β),

(xvi) (γ) gives almost the same result as (x) (γ),

(xvi) (δ) is not quite so good as ... ... (x) (δ).

So with the exception of (α) the extreme ordinate formulae are slightly better than the mid-ordinate formulae with the same number of elements of area. This may be due to the fact that in the former case we are using one ordinate more.

As another example we take $\int_0^3 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$, using 16 ordinates. We have

x	z	δ^2	δ^4	δ^6
·09375	·3971929			
·28125	·3834718	− ·0123157		
·46875	·3574350	− ·0097416	·0007494	
·65625	·3216566	− ·0064181	·0002426	·0000193
·84375	·2794601	− ·0028520	− ·0002449	·0001092
1·03125	·2344116	·0004692	− ·0006232	·0001616
1·21875	·1898323	·0031672	− ·0008399	·0001725
1·40625	·1484202	·0050253	− ·0008841	·0001415
1·59375	·1120334	·0059993	− ·0007868	·0000932
1·78125	·0816459	·0061865	− ·0005963	·0000307
1·96875	·0574449	·0057774	− ·0003751	− ·0000124
2·15625	·0390213	·0049932	− ·0001663	− ·0000455
2·34375	·0255909	·0040427	− ·0000030	− ·0000557
2·53125	·0162032	·0030892	+ ·0001046	
2·71875	·0099047	·0022403		
2·90625	·0058465			

All the differences are given, though not required in the calculation, as their examination affords a check on the work. We have

$$
\begin{array}{rlrl}
m = 16 & & & \\
z_{\frac{1}{2}} + z_{\frac{3}{2}} + \dots + z_{m-\frac{3}{2}} & & = & 2{\cdot}6595713 \\
2(\delta^2 z_{\frac{3}{2}} + \delta^2 z_{m-\frac{3}{2}}) = & -\ {\cdot}0201508 & & \\
-(\delta^2 z_{\frac{5}{2}} + \delta^2 z_{m-\frac{5}{2}}) = & {\cdot}0066524 & & \\
(z_{\frac{1}{2}} - z_{\frac{3}{2}} - z_{m-\frac{3}{2}} + z_{m-\frac{1}{2}}) = & {\cdot}0096629 & & \\
 & -\ {\cdot}0038355\,(\div 24) & = - & {\cdot}0001598 \\
395(\delta^4 z_{\frac{5}{2}} + \delta^4 z_{m-\frac{5}{2}}) = & {\cdot}3373300 & & \\
-189(\delta^4 z_{\frac{7}{2}} + \delta^4 z_{m-\frac{7}{2}}) = & -\ {\cdot}0452844 & & \\
-17(\delta^2 z_{\frac{3}{2}} - \delta^2 z_{\frac{5}{2}} - \delta^2 z_{m-\frac{5}{2}} + \delta^2 z_{m-\frac{3}{2}}) = & {\cdot}0581910 & & \\
 & {\cdot}3502366\,(\div 5760) & = & {\cdot}0000608 \\
58239(\delta^6 z_{\frac{7}{2}} + \delta^6 z_{m-\frac{7}{2}}) = & -2{\cdot}1198996 & & \\
-28242(\delta^6 z_{\frac{9}{2}} + \delta^6 z_{m-\frac{9}{2}}) = & -1{\cdot}8187848 & & \\
367(\delta^4 z_{\frac{5}{2}} - \delta^4 z_{\frac{7}{2}} - \delta^4 z_{m-\frac{7}{2}} + \delta^4 z_{m-\frac{5}{2}}) = & {\cdot}2254848 & & \\
 & -3{\cdot}7131996\,(\div 967680) & = - & {\cdot}0000038 \\
 & & & 2{\cdot}6594685 \\
 & & h = & {\cdot}1875 \\
 & \int_0^3 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dx & = & {\cdot}4986503
\end{array}
$$

This result is in excess by ·0000002 of the true value.

Summary of Part I

The reader may perhaps be somewhat embarrassed by the number of alternative formulae so far given, nor is it easy to give any general rules as to which to use. For our own part we favour the following three:

(A) Weddle's Formula ((ζ) p. 5).

(B) Gauss' Formula (p. 8).

(C) A Central Difference Formula ((xv) or (ix) or their alternative ordinate forms (xvi) (δ) and (x) (δ)).

A large experience has shown that Weddle's formula gives excellent results for continuous mathematical functions. Gauss' method gives remarkable accuracy for the number of ordinates employed, though their calculation at unequal intervals may be troublesome, while the central difference formulae have the very great advantage that we may work with differences and can see, almost at a glance, what terms may and what may not be neglected (although if we have no doubts on that point and have a calculating machine

ıt our disposal, it is probably rather quicker to use their ordinate forms). It s probable that formulae (ϵ), (ι), (μ), (ξ) and (π) give results not inferior to hese three formulae, but of the former we have not so much experience.

In making these remarks we have had mathematical functions principally n mind. When dealing with rough data, it must be remembered that the errors will make the higher differences erratic and it will consequently be better to use a formula such as Parmentier's which does not involve the higher differences. But considerations of this nature would seem more properly to belong to the subject of smoothing.

PART II. CUBATURE

A. Introductory

We now proceed to the consideration of cubature formulae or the evaluation of double integrals.

The most general form of double integral may be written

$$z = \int_{x=a}^{x=b} \int_{y=\phi_1(x)}^{y=\phi_2(x)} f(x, y)\, dx dy,$$

and in order to calculate this by quadratures the most obvious method of procedure seems to be to take a convenient series of equidistant values of x: $a = a_1, a_2, a_3 \ldots a_n = b$, and by a single quadrature to calculate each of the integrals $z_r = \int_{y=\phi_1(a_r)}^{y=\phi_2(a_r)} f(a_r, y)\, dy$ for all the n values of r, and finally, by another univariate quadrature to calculate $\int_a^b z_x dx$. This process needs the performance of $n+1$ quadratures, the actual univariate quadrature formula to be employed and the number of ordinates to be taken varying with the particular circumstances of the case and the accuracy required. We shall give an example of this process later; in many cases it may be the only possible one, but the main object of this paper is to discuss a more general treatment, applicable in all cases where the function has continuous curvature and the limits of integration are constants and in a number of other cases besides.

We shall deduce, from bivariate central difference interpolation formulae, cubature formulae by which the approximate volume is directly expressible in terms of certain equidistant ordinates. It is obvious that we have a certain freedom in the choice of ordinates on which to base the cubature.

If we are calculating $\int_0^{mh}\int_0^{nk} z\,dx\,dy$ we may suppose the volume to be divided into mn portions or prisms by m vertical planes at intervals h in one direction, and n at intervals k in the other.

Then, as Dr Sheppard has pointed out*, we might use as systems of ordinates:

(i) The corner ordinates of the constituent prisms.

(ii) The middle ordinates of the faces of the prisms in one direction.

(iii) The central ordinates of the prisms.

We shall consider both (i) and (iii) but not (ii) as it is inconvenient to have the ordinates differently placed in one direction from the other.

We now turn to the establishment of the formulae.

B. Extreme Ordinate and Mid-ordinate Cubature Formulae for $\int_0^{\infty}\int_{-\infty}^{\infty} z\,dx\,dy$

We start from Pearson's mid-panel bivariate central difference interpolation formula†. Writing $\phi = 1-\theta$, $\psi = 1-\chi$, we have:

$$\begin{aligned}
z_{\theta,\chi} = {} & (1-\theta)(1-\chi)z_{00} + (1-\theta)\chi z_{01} + \theta(1-\chi)z_{10} + \theta\chi z_{11} \\
& - \tfrac{1}{6}\theta(1-\theta)\{(2-\theta)[(1-\chi)\delta^2 z_{00} + \chi\delta^2 z_{01}] + (1+\theta)[(1-\chi)\delta^2 z_{10} + \chi\delta^2 z_{11}]\} \\
& - \tfrac{1}{6}\chi(1-\chi)\{(2-\chi)[(1-\theta)\delta'^2 z_{00} + \theta\delta'^2 z_{10}] + (1+\chi)[(1-\theta)\delta'^2 z_{01} + \theta\delta'^2 z_{11}]\} \\
& + \tfrac{1}{120}\theta(1-\theta)(1+\theta)(2-\theta)\{(3-\theta)[(1-\chi)\delta^4 z_{00} + \chi\delta^4 z_{01}] \\
& \qquad + (2+\theta)[(1-\chi)\delta^4 z_{10} + \chi\delta^4 z_{11}]\} \\
& + \tfrac{1}{36}\theta(1-\theta)\chi(1-\chi)\{(2-\theta)(2-\chi)\delta^2\delta'^2 z_{00} + (2-\theta)(1+\chi)\delta^2\delta'^2 z_{01} \\
& \qquad + (1+\theta)(2-\chi)\delta^2\delta'^2 z_{10} + (1+\theta)(1+\chi)\delta^2\delta'^2 z_{11}\} \\
& + \tfrac{1}{120}\chi(1-\chi)(1+\chi)(2-\chi)\{(3-\chi)[(1-\theta)\delta'^4 z_{00} + \theta\delta'^4 z_{10}] \\
& \qquad + (2+\chi)[(1-\theta)\delta'^4 z_{01} + \theta\delta'^4 z_{11}]\} \\
& - \tfrac{1}{720}\theta(1-\theta)\chi(1-\chi)(1+\theta)(2-\theta)\{(3-\theta)[(2-\chi)\delta^4\delta'^2 z_{00} + (1+\chi)\delta^4\delta'^2 z_{01}] \\
& \qquad + (2+\theta)[(2-\chi)\delta^4\delta'^2 z_{10} + (1+\chi)\delta^4\delta'^2 z_{11}]\} \\
& - \tfrac{1}{720}\theta(1-\theta)\chi(1-\chi)(1+\chi)(2-\chi)\{(3-\chi)[(2-\theta)\delta^2\delta'^4 z_{00} + (1+\theta)\delta^2\delta'^4 z_{10}] \\
& \qquad + (2+\chi)[(2-\theta)\delta^2\delta'^4 z_{01} + (1+\theta)\delta^2\delta'^4 z_{11}]\} \\
& - \tfrac{1}{5040}\theta(1-\theta)(2-\theta)(3-\theta)(1+\theta)(2+\theta)\{(4-\theta)[(1-\chi)\delta^6 z_{00} + \chi\delta^6 z_{01}] \\
& \qquad + (3+\theta)[(1-\chi)\delta^6 z_{10} + \chi\delta^6 z_{11}]\} \\
& - \tfrac{1}{5040}(1-\chi)\chi(2-\chi)(3-\chi)(1+\chi)(2+\chi)\{(4-\chi)[(1-\theta)\delta'^6 z_{00} + \theta\delta'^6 z_{10}] \\
& \qquad + (3+\chi)[(1-\theta)\delta'^6 z_{01} + \theta\delta'^6 z_{11}]\} \\
& + \text{terms of eighth order} \ldots\ldots\ldots\ldots \text{(i).}
\end{aligned}$$

* *Proceedings of the London Mathematical Society*, Vol. XXXII, 1900, 'Some Quadrature Formulae.'

† *Tracts for Computers*, III, p. 8.

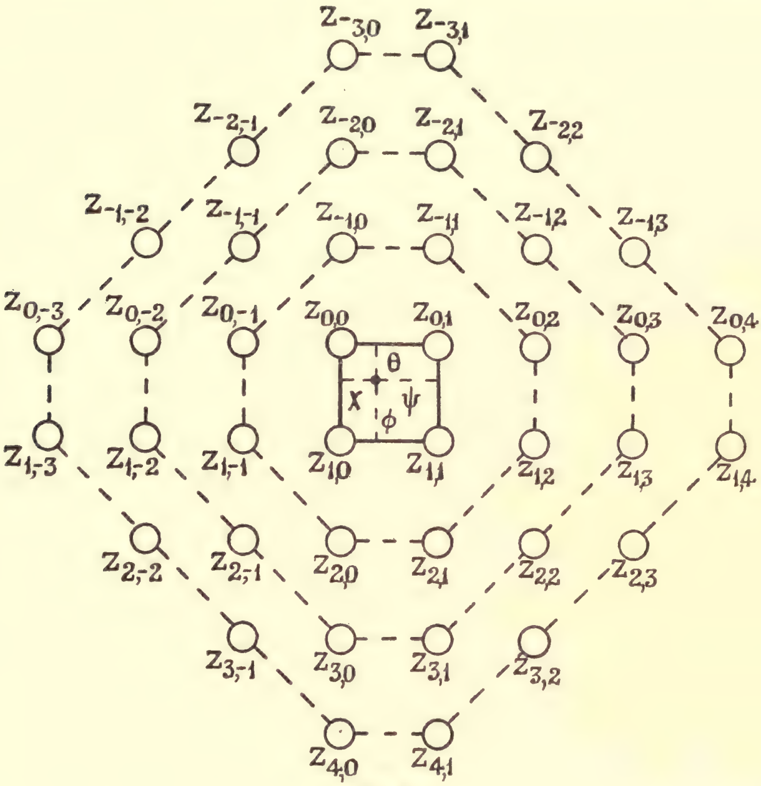


Fig. 2

We now evaluate $$\int_0^1 \int_0^1 z_{\theta,\chi}\, d\theta\, d\chi.$$

Let Z_r denote the terms with differences of apparent order r in $z_{\theta,\chi}$. Then

$$\begin{aligned}\int_0^1\int_0^1 Z_0\, d\theta\, d\chi &= \int_0^1 d\theta \Big[(\chi - \tfrac{1}{2}\chi^2)(1-\theta)\, z_{00} + \tfrac{1}{2}\chi^2(1-\theta)\, z_{01} \\ &\qquad + \theta(\chi - \tfrac{1}{2}\chi^2)\, z_{10} + \theta\tfrac{1}{2}\chi^2 z_{11}\Big]_{\chi=0}^{\chi=1} \\ &= \int_0^1 d\theta\, [\tfrac{1}{2}(1-\theta)\, z_{00} + \tfrac{1}{2}(1-\theta)\, z_{01} + \tfrac{1}{2}\theta z_{10} + \tfrac{1}{2}\theta z_{11}] \\ &= \tfrac{1}{4}(z_{00} + z_{01} + z_{10} + z_{11}) \text{ ....(ii).}\end{aligned}$$

$$\begin{aligned}\int_0^1\int_0^1 Z_2\, d\theta\, d\chi &= \int_0^1 d\theta \Big[-\tfrac{1}{6}\theta(1-\theta)(2-\theta)\,[(\chi - \tfrac{1}{2}\chi^2)\,\delta^2 z_{00} + \tfrac{1}{2}\chi^2\delta^2 z_{01}] \\ &\qquad - \tfrac{1}{6}\theta(1-\theta^2)\,[(\chi - \tfrac{1}{2}\chi^2)\,\delta^2 z_{10} + \tfrac{1}{2}\chi^2\delta^2 z_{11}] \\ &\qquad - \tfrac{1}{6}(\chi^2 - \chi^3 + \tfrac{1}{4}\chi^4)\,[(1-\theta)\,\delta'^2 z_{00} + \theta\delta'^2 z_{10}] \\ &\qquad - \tfrac{1}{6}\left(\frac{\chi^2}{2} - \frac{\chi^4}{4}\right)[(1-\theta)\,\delta'^2 z_{01} + \theta\delta'^2 z_{11}]\Big]_{\chi=0}^{\chi=1} \\ &= \int_0^1 d\theta\, [-\tfrac{1}{12}\theta(1-\theta)(2-\theta)[\delta^2 z_{00} + \delta^2 z_{01}] - \tfrac{1}{12}\theta(1-\theta^2)[\delta^2 z_{10} + \delta^2 z_{11}] \\ &\qquad - \tfrac{1}{24}\{(1-\theta)(\delta'^2 z_{00} + \delta'^2 z_{01}) + \theta(\delta'^2 z_{10} + \delta'^2 z_{11})\}] \\ &= -\tfrac{1}{48}(\delta^2 z_{00} + \delta^2 z_{01} + \delta^2 z_{10} + \delta^2 z_{11} + \delta'^2 z_{00} + \delta'^2 z_{01} + \delta'^2 z_{10} + \delta'^2 z_{11}) \text{...(iii).}\end{aligned}$$

$$
\begin{aligned}
\int_0^1\int_0^1 Z_4\,d\theta\,d\chi = \int_0^1 d\theta\Big[& \tfrac{1}{120}\theta(1-\theta^2)(2-\theta)\{(3-\theta)[(\chi-\tfrac{1}{2}\chi^2)\delta^4 z_{00}+\tfrac{1}{2}\chi^2\delta^4 z_{01}] \\
& \qquad +(2+\theta)[(\chi-\tfrac{1}{2}\chi^2)\delta^4 z_{10}+\tfrac{1}{2}\chi^2\delta^4 z_{11}]\} \\
& +\tfrac{1}{36}\theta(1-\theta)(2-\theta)(\chi^2-\chi^3+\tfrac{1}{4}\chi^4)\delta^2\delta'^2 z_{00} \\
& +\tfrac{1}{36}\theta(1-\theta)(2-\theta)\left(\frac{\chi^2}{2}-\frac{\chi^4}{4}\right)\delta^2\delta'^2 z_{01} \\
& +\tfrac{1}{36}\theta(1-\theta^2)(\chi^2-\chi^3+\tfrac{1}{4}\chi^4)\delta^2\delta'^2 z_{10} \\
& +\tfrac{1}{36}\theta(1-\theta^2)\left(\frac{\chi^2}{2}-\frac{\chi^4}{4}\right)\delta^2\delta'^2 z_{11} \\
& +\tfrac{1}{120}\left(3\chi^2-\frac{5\chi^3}{3}-\frac{5\chi^4}{4}+\chi^5-\frac{\chi^6}{6}\right)[(1-\theta)\delta'^4 z_{00}+\theta\delta'^4 z_{10}] \\
& +\tfrac{1}{120}\left(2\chi^2-\frac{5\chi^4}{4}+\frac{\chi^6}{6}\right)[(1-\theta)\delta'^4 z_{01}+\theta\delta'^4 z_{11}]\Big]_{\chi=0}^{\chi=1} \\
= \int_0^1 d\theta\,[& \tfrac{1}{240}\theta(1-\theta^2)(2-\theta)(3-\theta)[\delta^4 z_{00}+\delta^4 z_{01}] \\
& \qquad +\tfrac{1}{240}\theta(1-\theta^2)(4-\theta^2)[\delta^4 z_{10}+\delta^4 z_{11}] \\
& +\tfrac{1}{144}\theta(2-3\theta+\theta^2)\delta^2\delta'^2 z_{00}+\tfrac{1}{144}\theta(2-3\theta+\theta^2)\delta^2\delta'^2 z_{01} \\
& +\tfrac{1}{144}\theta(1-\theta^2)\delta^2\delta'^2 z_{10}+\tfrac{1}{144}\theta(1-\theta^2)\delta^2\delta'^2 z_{11} \\
& +\tfrac{11}{1440}[(1-\theta)\delta'^4 z_{00}+\theta\delta'^4 z_{10}+(1-\theta)\delta'^4 z_{01}+\theta\delta'^4 z_{11}]] \\
= \tfrac{11}{2880} & (\delta^4 z_{00}+\delta^4 z_{01}+\delta^4 z_{10}+\delta^4 z_{11}+\delta'^4 z_{00}+\delta'^4 z_{01}+\delta'^4 z_{10}+\delta'^4 z_{11}) \\
& +\tfrac{1}{576}(\delta^2\delta'^2 z_{00}+\delta^2\delta'^2 z_{01}+\delta^2\delta'^2 z_{10}+\delta^2\delta'^2 z_{11})\ldots\ldots\ldots\text{(iv)}.
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1\int_0^1 Z_6\,d\theta\,d\chi = \int_0^1 d\theta\Big[& -\tfrac{1}{720}\theta(1-\theta^2)(2-\theta)(3-\theta)(\chi^2-\chi^3+\tfrac{1}{4}\chi^4)\delta^4\delta'^2 z_{00} \\
& -\tfrac{1}{720}\theta(1-\theta^2)(2-\theta)(3-\theta)\left(\frac{\chi^2}{2}-\frac{\chi^4}{4}\right)\delta^4\delta'^2 z_{01} \\
& -\tfrac{1}{720}\theta(1-\theta^2)(4-\theta^2)(\chi^2-\chi^3+\tfrac{1}{4}\chi^4)\delta^4\delta'^2 z_{10} \\
& -\tfrac{1}{720}\theta(1-\theta^2)(4-\theta^2)\left(\frac{\chi^2}{2}-\frac{\chi^4}{4}\right)\delta^4\delta'^2 z_{11} \\
& -\tfrac{1}{720}\theta(1-\theta)(2-\theta)\left(3\chi^2-\frac{5\chi^3}{3}-\frac{5\chi^4}{4}+\chi^5-\frac{\chi^6}{6}\right)\delta^2\delta'^4 z_{00} \\
& -\tfrac{1}{720}\theta(1-\theta^2)\left(3\chi^2-\frac{5\chi^3}{3}-\frac{5\chi^4}{4}+\chi^5-\frac{\chi^6}{6}\right)\delta^2\delta'^4 z_{10} \\
& -\tfrac{1}{720}\theta(1-\theta)(2-\theta)\left(2\chi^2-\frac{5\chi^4}{4}+\frac{\chi^6}{6}\right)\delta^2\delta'^4 z_{01} \\
& -\tfrac{1}{720}\theta(1-\theta^2)\left(2\chi^2-\frac{5\chi^4}{4}+\frac{\chi^6}{6}\right)\delta^2\delta'^4 z_{11} \\
& -\tfrac{1}{5040}\theta(1-\theta^2)(4-\theta^2)(3-\theta) \\
& \qquad \times[(4-\theta)\{(\chi-\tfrac{1}{2}\chi^2)\delta^6 z_{00}+\tfrac{1}{2}\chi^2\delta^6 z_{01}\} \\
& \qquad +(3+\theta)\{(\chi-\tfrac{1}{2}\chi^2)\delta^6 z_{10}+\tfrac{1}{2}\chi^2\delta^6 z_{11}\}] \\
& -\tfrac{1}{5040}[(1-\theta)\delta'^6 z_{00}+\theta\delta'^6 z_{10}] \\
& \qquad \times\left(24\chi^2-\frac{28\chi^3}{3}-14\chi^4+7\chi^5+\tfrac{7}{6}\chi^6-\chi^7+\tfrac{1}{8}\chi^8\right) \\
& -\tfrac{1}{5040}[(1-\theta)\delta'^6 z_{01}+\theta\delta'^6 z_{11}] \\
& \qquad \times(18\chi^2-\tfrac{49}{4}\chi^4+\tfrac{7}{3}\chi^6-\tfrac{1}{8}\chi^8)\Big]_{\chi=0}^{\chi=1}
\end{aligned}
$$

$$
\begin{aligned}
=\int_0^1 d\theta \{ & -\tfrac{1}{2880}\theta(1-\theta^2)(2-\theta)(3-\theta)(\delta^4\delta'^2 z_{00}+\delta^4\delta'^2 z_{01}) \\
& -\tfrac{1}{2880}\theta(1-\theta^2)(4-\theta^2)(\delta^4\delta'^2 z_{10}+\delta^4\delta'^2 z_{11}) \\
& -\tfrac{11}{8640}\theta(1-\theta)(2-\theta)(\delta^2\delta'^4 z_{00}+\delta^2\delta'^4 z_{01}) \\
& -\tfrac{11}{8640}\theta(1-\theta^2)(\delta^2\delta'^4 z_{10}+\delta^2\delta'^4 z_{11}) \\
& -\tfrac{1}{10080}\theta(1-\theta^2)(4-\theta^2)(3-\theta)(4-\theta)(\delta^6 z_{00}+\delta^6 z_{01}) \\
& -\tfrac{1}{10080}\theta(1-\theta^2)(4-\theta^2)(9-\theta^2)(\delta^6 z_{00}+\delta^6 z_{11}) \\
& -\tfrac{191}{120960}[(1-\theta)(\delta'^6 z_{00}+\delta'^6 z_{01})+\theta(\delta'^6 z_{10}+\delta'^6 z_{11}]\} \\
=-\tfrac{1}{241920}\{ & 77(\delta^4\delta'^2+\delta^2\delta'^4)(z_{00}+z_{01}+z_{10}+z_{11}) \\
& +191(\delta^6+\delta'^6)(z_{00}+z_{01}+z_{10}+z_{11})\} \quad \ldots\text{(v)}.
\end{aligned}
$$

Combining (ii), (iii), (iv), (v) we reach

$$
\begin{aligned}
\int_0^1\int_0^1 z_{\theta,\chi}\, d\theta d\chi = & \tfrac{1}{4}(z_{00}+z_{01}+z_{10}+z_{11})-\tfrac{1}{48}(\delta^2+\delta'^2)(z_{00}+z_{01}+z_{10}+z_{11}) \\
& +\tfrac{1}{2880}\{11(\delta^4+\delta'^4)+5\delta^2\delta'^2\}(z_{00}+z_{01}+z_{10}+z_{11}) \\
-\tfrac{1}{241920}\{77(\delta^4\delta'^2+\delta^2\delta'^4)&(z_{00}+z_{01}+z_{10}+z_{11})+191(\delta^6+\delta'^6)(z_{00}+z_{01}+z_{10}+z_{11})\} \\
& +\text{terms of eighth order};
\end{aligned}
$$

and if the ordinates of the surface form a rectangular network of meshes h by k, the volume over the rectangle (00, 01, 10, 11)

$$
\begin{aligned}
=hk[& \tfrac{1}{4}(z_{00}+z_{01}+z_{10}+z_{11})-\tfrac{1}{48}(\delta^2+\delta'^2)(z_{00}+z_{01}+z_{10}+z_{11}) \\
& +\tfrac{1}{2880}\{11(\delta^4+\delta'^4)+5\delta^2\delta'^2\}(z_{00}+z_{01}+z_{10}+z_{11}) \\
& -\tfrac{1}{241920}\{77(\delta^4\delta'^2+\delta^2\delta'^4)(z_{00}+z_{01}+z_{10}+z_{11}) \\
& \qquad +191(\delta^6+\delta'^6)(z_{00}+z_{01}+z_{10}+z_{11})\}] \quad \ldots\text{(vi)}.
\end{aligned}
$$

We will now express (vi) in terms of z's only, including

(α) no terms with higher order differences than first,
(β) „ „ „ „ „ third,
(γ) „ „ „ „ „ fifth,
(δ) „ „ „ „ „ seventh.

This can be done by means of the relations between the operators,

$$\delta^2=\left(\sqrt{E}-\frac{1}{\sqrt{E}}\right)^2=E-2+\frac{1}{E},$$

$$\delta'^2=\left(\sqrt{E'}-\frac{1}{\sqrt{E'}}\right)^2=E'-2+\frac{1}{E'},$$

whence

$$\delta^4=E^2-4E+6-\frac{4}{E}+\frac{1}{E^2},$$

$$\delta'^4=E'^2-4E'+6-\frac{4}{E'}+\frac{1}{E'^2},$$

$$\delta^2\delta'^2=\left(EE'+\frac{1}{EE'}\right)-2\left(E+\frac{1}{E}+E'+\frac{1}{E'}\right)+\frac{E}{E'}+\frac{E'}{E}+4.$$

$$\delta^6 = E^3 - 6E^2 + 15E - 20 + \frac{15}{E} - \frac{6}{E^2} + \frac{1}{E^3},$$

$$\delta'^6 = E'^3 - 6E'^2 + 15E' - 20 + \frac{15}{E'} - \frac{6}{E'^2} + \frac{1}{E'^3},$$

$$\delta^4\delta'^2 = \left(E^2E' + \frac{1}{E^2E'}\right) - 4\left(EE' + \frac{1}{EE'}\right) + 6\left(E' + \frac{1}{E'}\right) - 4\left(\frac{E'}{E} + \frac{E}{E'}\right) + \frac{E'}{E^2} + \frac{E^2}{E'} - 2\left(E^2 - 4E + 6 - \frac{4}{E} + \frac{1}{E^2}\right),$$

$$\delta'^4\delta^2 = \left(E'^2E + \frac{1}{E'^2E}\right) - 4\left(EE' + \frac{1}{EE'}\right) + 6\left(E + \frac{1}{E}\right) - 4\left(\frac{E'}{E} + \frac{E}{E'}\right) + \frac{E}{E'^2} + \frac{E'^2}{E} - 2\left(E'^2 - 4E' + 6 - \frac{4}{E'} + \frac{1}{E'^2}\right).$$

After some reductions the following expressions are obtained for the volume of the element founded by the ordinates z_{00}, z_{01}, z_{10}, z_{11}.

$$\left\{\begin{aligned}
&(\alpha)\quad \frac{hk}{4}(z_{00}+z_{01}+z_{10}+z_{11}),\\
&(\beta)\quad \frac{hk}{48}[14(z_{00}+z_{01}+z_{10}+z_{11})-(z_{-10}+z_{-11}+z_{02}+z_{12}+z_{21}+z_{20}+z_{1-1}+z_{0-1})],\\
&(\gamma)\quad \frac{hk}{2880}[889(z_{00}+z_{01}+z_{10}+z_{11})-98(z_{-10}+z_{-11}+z_{02}+z_{12}+z_{21}+z_{20}+z_{1-1}+z_{0-1})\\
&\qquad +5(z_{-1-1}+z_{-12}+z_{22}+z_{2-1})\\
&\qquad +11(z_{-20}+z_{-21}+z_{03}+z_{13}+z_{31}+z_{30}+z_{1-2}+z_{0-2})],\\
&(\delta)\quad \frac{hk}{241920}[76894(z_{00}+z_{01}+z_{10}+z_{11})\\
&\qquad -10336(z_{-10}+z_{0-1}+z_{1-1}+z_{20}+z_{21}+z_{12}+z_{02}+z_{-11})\\
&\qquad +882(z_{-1-1}+z_{-12}+z_{22}+z_{2-1})\\
&\qquad +1956(z_{-20}+z_{-21}+z_{03}+z_{13}+z_{31}+z_{30}+z_{1-2}+z_{0-2})\\
&\qquad -77(z_{-22}+z_{-13}+z_{23}+z_{32}+z_{3-1}+z_{2-2}+z_{-1-2}+z_{-2-1})\\
&\qquad -191(z_{-30}+z_{0-3}+z_{1-3}+z_{40}+z_{41}+z_{14}+z_{04}+z_{-31})]\ldots\ldots\text{(vii)}.
\end{aligned}\right.$$

(vii) α, β, γ, δ are respectively correct, up to and including 1st, 3rd, 5th and 7th order differences. It will be seen that the feet of those ordinates which have the same coefficient are in every case equidistant from the centre of the square (00, 01, 10, 11).

It is clear that any double integral of the type $\int_{ph}^{qh}\int_{rk}^{sk} z\,dx\,dy$ could be approximately evaluated by summing formulae of the type (vii) for each panel within the area over which the function is to be integrated; but this would be open to the objection that the resulting cubature formulae would contain ordinates outside the area, which would be inconvenient for practical purposes.

This difficulty will be avoided later by establishing special formulae for the panels on the boundary of the rectangular area; but there is one case which may profitably be deduced from the formulae (vii), which we now proceed to consider.

Suppose $\int_0^\infty \int_{-\infty}^\infty z\,dx\,dy$ to exist, then approximations to its value may be obtained by summing the formulae (vii) for every panel in the half-plane under consideration.

Let

$$u_r = \ldots + z_{r,-s} + z_{r,-s+1} + \ldots + z_{r,-1} + z_{r,0} + z_{r,1} + \ldots + z_{r,s-1} + z_{r,s} + \ldots,$$

or we define u_r to be the sum of the ordinates along the line $x = rh$. Then summing over the half-plane we have

$$\Sigma\,(z_{00} + z_{01} + z_{10} + z_{11}) = 2u_0 + 4\,(u_1 + u_2 + u_3 + \ldots),$$

$$\Sigma\,(z_{-10} + z_{-11} + z_{02} + z_{12} + z_{21} + z_{20} + z_{1-1} + z_{0-1})$$
$$= 2u_{-1} + 4u_0 + 6u_1 + 8\,(u_2 + u_3 + u_4 + \ldots),$$

$$\Sigma\,(z_{-1-1} + z_{-12} + z_{2-1} + z_{22}) = 2u_{-1} + 2u_0 + 2u_1 + 4\,(u_2 + u_3 + \ldots),$$

$$\Sigma\,(z_{-20} + z_{-21} + z_{03} + z_{13} + z_{31} + z_{30} + z_{1-2} + z_{0-2})$$
$$= 2u_{-2} + 2u_{-1} + 4u_0 + 6u_1 + 6u_2 + 8\,(u_3 + u_4 + \ldots),$$

$$\Sigma\,(z_{-22} + z_{-13} + z_{23} + z_{32} + z_{3-1} + z_{2-2} + z_{-1-2} + z_{-2-1})$$
$$= 2u_{-2} + 4u_{-1} + 4u_0 + 4u_1 + 6u_2 + 8\,(u_3 + u_4 + \ldots),$$

$$\Sigma\,(z_{-30} + z_{-31} + z_{04} + z_{14} + z_{41} + z_{40} + z_{1-3} + z_{0-3})$$
$$= 2u_{-3} + 2u_{-2} + 2u_{-1} + 4u_0 + 6u_1 + 6u_2 + 6u_3 + 8\,(u_4 + \ldots).$$

Substituting these results in (vii) we obtain the following approximations* for $\int_0^\infty \int_{-\infty}^\infty z\,dx\,dy$:

$$\left\{\begin{array}{ll} (\alpha) & \dfrac{hk}{2}\,[u_0 + 2\,(u_1 + u_2 + u_3 + u_4 + \ldots)], \\[2ex] (\beta) & \dfrac{hk}{24}\,[-u_{-1} + 12u_0 + 25u_1 + 24\,(u_2 + u_3 + u_4 + \ldots)], \\[2ex] (\gamma) & \dfrac{hk}{1440}\,[11u_{-2} - 82u_{-1} + 720u_0 + 1522u_1 + 1429u_2 + 1440\,(u_3 + u_4 + \ldots)], \\[2ex] (\delta) & \dfrac{hk}{120960}\,[-191u_{-3} + 1688u_{-2} - 7843u_{-1} + 60480u_0 + 128803u_1 \\ & \qquad + 119272u_2 + 121151u_3 + 120960\,(u_4 + u_5 + u_6 + \ldots)] \ldots \text{(viii)}. \end{array}\right.$$

* These formulae, treated as univariate formulae for the u's, are, to the order of differences to which they go, identical with formulae (x) of Part I and might have been derived directly from them, but we have preferred to obtain them from the bivariate interpolation formulae rather than to quote previous results. Formulae (x) of Part I must be used on the u's if the ordinates on the negative side of the origin are unknown.

These formulae are respectively correct up to and including 1st, 3rd, 5th and 7th order differences.

We must now note a rather remarkable fact. Suppose the surface under consideration to be symmetrical about the plane $y=0$. We have

$$u_{-1}=u_1, \qquad u_{-2}=u_2, \qquad u_{-3}=u_3,$$

and substituting these relations in (viii) (α), (β), (γ), (δ) each reduces to

$$hk\left[\tfrac{1}{2}u_0+u_1+u_2+u_3+u_4+\ldots\right] \ldots\ldots\ldots\ldots\ldots\ldots\text{(ix)}.$$

This at first sight seems surprising, as it suggests that for the case of a symmetrical surface and the particular double integral we are considering, we should always reach formula (ix), no matter to what order of differences we proceeded in our original formula; or in other words that all differences beyond the first vanish in the summation. It will now be proved that this is the case.

Let $\qquad z=f(x, y)$ and $v_x=\int_{-\infty}^{\infty} f(x, y)\,dy,$

then $\qquad \int_0^{\infty}\int_{-\infty}^{\infty} z\,dx\,dy=\int_0^{\infty} v_x dx.$

Using the Euler-McLaurin formula we have

$$\int_0^{\infty}\int_{-\infty}^{\infty} z\,dx\,dy=h\left(\tfrac{1}{2}v_0+v_1+v_2+v_3+\ldots\right)-\left[\tfrac{1}{12}h^2\frac{dv_x}{dx}-\tfrac{1}{720}h^4\frac{d^3v_x}{dx^3}+\ldots\right]_0^{\infty},$$

$$v_r=k\left(\ldots+z_{r,-s}+z_{r,-s+1}+\ldots+z_{r,-1}+z_{r,0}+z_{r,1}+\ldots+z_{r,s-1}+z_{r,s}+\ldots\right)$$
$$-\left[\tfrac{1}{12}k^2\frac{dz_{r,y}}{dy}-\tfrac{1}{720}k^4\frac{d^3z_{r,y}}{dy^3}-\ldots\right]_{-\infty}^{\infty} \ldots\text{(x)}.$$

Now since v_x is a continuous function and is symmetrical about the origin, all its odd differential coefficients vanish* there, and since it has high contact at infinity†, all its differential coefficients will vanish at infinity.

Again $z_{r,y}$ must have high contact at $-\infty$ and ∞ †, and so its differential coefficients will vanish at both ends of its range.

* In the neighbourhood of the origin v_x can be expressed in the form

$$v_x=v_0+v_0'x+v_0''\frac{x^2}{\underline{|2}}+v_0'''\frac{x^3}{\underline{|3}}+\ldots,$$

and since it is symmetrical $\qquad v_x=v_0-v_0'x+v_0''\dfrac{x^2}{\underline{|2}}-v_0'''\dfrac{x^3}{\underline{|3}}+\ldots,$

whence $\qquad v_0'=v_0'''=\text{etc.}=0.$

† Both v_x and $z_{r,y}$ are zero for x, y infinite, otherwise the double integral would not exist; and if any continuous function vanish at infinity its differential coefficients must also. For either (i) it is expressible for large values of the argument in the form

$$\frac{a_1}{x}+\frac{a_2}{x^2}+\ldots+\frac{a_p}{x^p}+\ldots,$$

And so all the differential coefficients in formula (x) vanish and we reach

$$\int_0^\infty \int_{-\infty}^\infty z\,dx\,dy = hk\left[\tfrac{1}{2}u_0 + u_1 + u_2 + u_3 + \dots\right] \quad \text{............(ix)},$$

the vanishing of the differential coefficients in (x) being equivalent to the vanishing of the differences when formulae (vi) are summed for the half-plane.

Accordingly, in the case of a symmetrical surface, (ix) is the only approximation we may obtain for the double integral under consideration, for the given distance apart of the ordinates*, and it may perhaps be anticipated that for this particular case it will be a good approximation.

1°. We now consider some examples:

$$\int_0^\infty \int_{-\infty}^\infty \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}\,dx\,dy.$$

The following table gives to nine decimal places the ordinates of the surface, calculated at unit intervals for x and y:

TABLE I. *Table of* $\frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$.

y	$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$	$\frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$					
0	·398942280	·159154943	·096532353	·021539279	·001768052	·000053390	·000000593
±1·0	·241970724	·096532353	·058549831	·013064233	·001072377	·000032383	·000000360
±2·0	·053990966	·021539279	·013064233	·002915024	·000239280	·000007226	·000000080
±3·0	·004431848	·001768052	·001072377	·000239280	·000019641	·000000593	·000000007
±4·0	·000133830	·000053390	·000032383	·000007226	·000000593	·000000018	
±5·0	·000001487	·000000593	·000000360	·000000080	·000000007		
	$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$	·398942280	·241970724	·053990966	·004431848	·000133830	·000001487
	x	0	1·0	2·0	3·0	4·0	5·0

or

$$\text{(ii)} \quad \lim_{x \to \infty} f(x)\,x^n = 0$$

for all values of n; in the latter case, it is easily proved that

$$\lim_{x \to \infty} f^r(x)\,x^n = 0$$

for all values of n, so that $f^r(x)$ is certainly 0 for x infinite; in the former case

$$f^r(x) = \sum_{p=1}^{\infty} (-1)^r \frac{p(p+1)\dots(p+r-1)}{x^{p+r}},$$

which is 0 for $x = \infty$.

* (ix) is not exact, for the Euler-McLaurin formula is only asymptotic. It would only be exact if the ordinates were infinitely close.

We find: $u_0 = \cdot 398942277$ $u_2 = \cdot 053990965$ $u_4 = \cdot 000133830$
$u_1 = \cdot 241970721$ $u_3 = \cdot 004431848$ $u_5 = \cdot 000001487$

whence using (ix) $\int_0^\infty \int_{-\infty}^\infty \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}\, dx\, dy = \cdot 499999990$

or we are in error by one in the eighth place. If we only take the ordinates to eight places we obtain ·49999997 which is correct to seven figures, and if only to seven places the result is ·4999998 which is correct to six figures. Clearly, we cannot expect our result to be accurate to more places than one less than the number of figures taken in the ordinates; we therefore conclude that the above distances apart of the ordinates are in this case adequate to give the result correct to seven figures.

We also tried the effect of increasing the number of ordinates while only using seven figures. Taking ordinates at every half unit we found ·4999998 for the volume, and taking them at every quarter unit gives the same result. The following conclusions are quite general:

(i) If we take the ordinates a given distance apart, no increase of the number of figures in each ordinate, beyond a certain maximum, will increase the accuracy of the result.

(ii) If we are working with a given number of figures in the ordinates, no increase of the number of ordinates, beyond a certain maximum, will increase the accuracy of the result.

In practice, the best method of procedure is to start with ordinates fairly wide apart, say at distances about equal to the standard deviation of the two variates under consideration and then to make a second computation taking the ordinates closer together. If the results of these two computations agree to r figures, the computer may feel confident that his result is correct to $r-1$ figures. Table II should convince the reader of the unnecessary labour that may result from taking the ordinates too close together.

2°. We now consider the computation of a double integral which not only illustrates formulae (viii), but also shows how it is sometimes possible, by a simple mathematical artifice, to reduce the integral to a form which admits of their employment, namely:

$$\int_{-\infty}^\infty \int_{-\infty}^x (1-\beta)\,\beta'^9\, dx\, dy^* \text{ where } \begin{cases} \beta = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx, \\ \beta' = \int_{-\infty}^y \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\, dy. \end{cases}$$

* $\frac{2\,\underline{|n}}{\underline{|n-p}\,\underline{|p}} \int_{-\infty}^\infty \int_{-\infty}^x (1-\beta)^p \beta'^{n-p}\, dx\, dy$ is the second moment about the origin of the frequency distribution of differences between the pth and $p+1$th individuals in samples of n taken from a 'normal' population of unit standard deviation.

By turning the axes through 45°, this integral may be expressed as $\int_0^\infty \int_{-\infty}^\infty z\,dxdy$; and therefore, if we are given the necessary numerical values of z, we may use formulae (viii). The process amounts to obtaining the 'u's,' by summing the ordinates along diagonals, instead of those in vertical columns (see Table III). Taking the ordinates at intervals of a quarter of a unit*, we have:

$u_{-3} = \cdot 7381118$	$u_2 = \cdot 0785199$	$u_7 = \cdot 0030897$	$u_{12} = \cdot 0000397$
$u_{-2} = \cdot 5066686$	$u_3 = \cdot 0447205$	$u_8 = \cdot 0014198$	$u_{13} = \cdot 0000144$
$u_{-1} = \cdot 3360149$	$u_4 = \cdot 0244448$	$u_9 = \cdot 0006230$	$u_{14} = \cdot 0000051$
$u_0 = \cdot 2149582$	$u_5 = \cdot 0128108$	$u_{10} = \cdot 0002614$	$u_{15} = \cdot 0000016$
$u_1 = \cdot 1326715$	$u_6 = \cdot 0064309$	$u_{11} = \cdot 0001044$	$u_{16} = \cdot 0000005$

Whence we obtain the following results for the double integral:

formula (viii) (β) ·0252603, (viii) (γ) ·0252705, (viii) (δ) ·0252708.

We conclude that to six figures the result is ·025271.

We now consider the cubature formulae which may be obtained from the Pearson two-dimensional mid-point interpolation formula; and shall show that these formulae are really identical with those that have already been obtained in (viii).

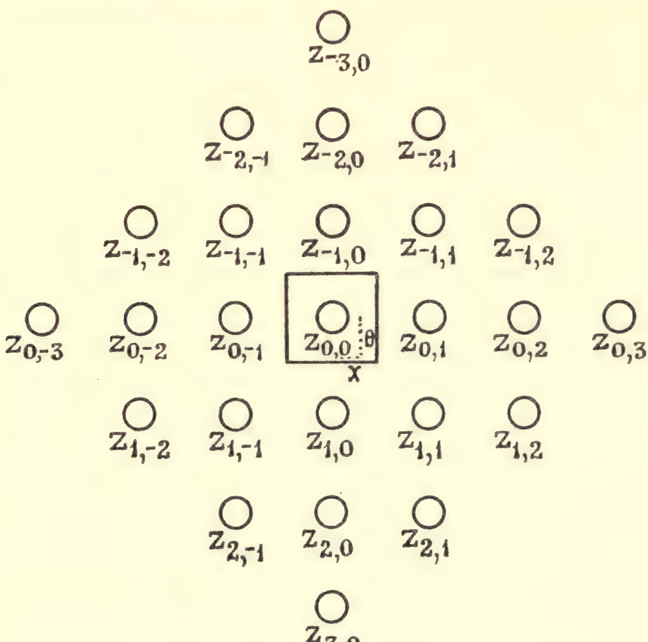


Fig. 3

* It was not found that half-unit intervals gave sufficiently accurate results.

The Pearson double, mid-point, interpolation formula is*:

$$z_{\theta,\chi} = z_{00} + \tfrac{1}{2}\chi(z_{01} - z_{0-1}) + \tfrac{1}{2}\theta(z_{10} - z_{-10}) + \tfrac{1}{4}\theta\chi(z_{11} - z_{1-1} - z_{-11} + z_{-1-1})] \quad \text{1st order}$$

$$\left.\begin{aligned}
&+ \tfrac{1}{2}\chi^2(1 - \tfrac{1}{2}\theta^2)\,\delta'^2 z_{00} + \tfrac{1}{2}\theta^2(1 - \tfrac{1}{2}\chi^2)\,\delta^2 z_{00} \\
&+ \tfrac{1}{8}\theta^2\chi^2(\delta'^2 z_{10} + \delta'^2 z_{-10} + \delta^2 z_{01} + \delta^2 z_{0-1})
\end{aligned}\right] \text{2nd order}$$

$$\left.\begin{aligned}
&+ \tfrac{1}{4}\theta^2\chi(\delta^2 z_{01} - \delta^2 z_{0-1}) + \tfrac{1}{4}\theta\chi^2(\delta'^2 z_{10} - \delta'^2 z_{-10}) \\
&- \tfrac{1}{12}\theta(1 - \theta^2)(\delta^2 z_{10} - \delta^2 z_{-10}) - \tfrac{1}{12}\chi(1 - \chi^2)(\delta'^2 z_{01} - \delta'^2 z_{0-1})
\end{aligned}\right] \text{3rd order}$$

$$\left.\begin{aligned}
&- \tfrac{1}{24}\theta\chi(1 - \theta^2)(\delta^2 z_{11} - \delta^2 z_{1-1} - \delta^2 z_{-11} + \delta^2 z_{-1-1}) \\
&- \tfrac{1}{24}\theta\chi(1 - \chi^2)(\delta'^2 z_{11} - \delta'^2 z_{1-1} - \delta'^2 z_{-11} + \delta'^2 z_{-1-1}) \\
&- \tfrac{1}{24}\theta^2(1 - \theta^2)\,\delta^4 z_{00} - \tfrac{1}{24}\chi^2(1 - \chi^2)\,\delta'^4 z_{00}
\end{aligned}\right] \text{4th order}$$

$$\left.\begin{aligned}
&- \tfrac{1}{48}\theta^2\chi(1 - \theta^2)(\delta^4 z_{01} - \delta^4 z_{0-1}) - \tfrac{1}{48}\theta\chi^2(1 - \chi^2)(\delta'^4 z_{10} - \delta'^4 z_{-10}) \\
&+ \tfrac{1}{240}\theta(1 - \theta^2)(4 - \theta^2)(\delta^4 z_{10} - \delta^4 z_{-10}) \\
&+ \tfrac{1}{240}\chi(1 - \chi^2)(4 - \chi^2)(\delta'^4 z_{01} - \delta'^4 z_{0-1}) \\
&- \tfrac{1}{24}\theta\chi^2(1 - \theta^2)(\delta^2\delta'^2 z_{10} - \delta^2\delta'^2 z_{-10}) \\
&- \tfrac{1}{24}\theta^2\chi(1 - \chi^2)(\delta^2\delta'^2 z_{01} - \delta^2\delta'^2 z_{0-1})
\end{aligned}\right] \text{5th order}$$

$$\left.\begin{aligned}
&+ \tfrac{1}{720}\theta^2(1 - \theta^2)(4 - \theta^2)\,\delta^6 z_{00} + \tfrac{1}{720}\chi^2(1 - \chi^2)(4 - \chi^2)\,\delta'^6 z_{00} \\
&- \tfrac{1}{48}\theta^2\chi^2(1 - \theta^2)\,\delta^4\delta'^2 z_{00} - \tfrac{1}{48}\theta^2\chi^2(1 - \chi^2)\,\delta^2\delta'^4 z_{00} \\
&+ \tfrac{1}{144}\theta\chi(1 - \theta^2)(1 - \chi^2)(\delta^2\delta'^2 z_{11} - \delta^2\delta'^2 z_{1-1} - \delta^2\delta'^2 z_{-11} + \delta^2\delta'^2 z_{-1-1}) \\
&+ \tfrac{1}{480}\theta\chi(1 - \theta^2)(4 - \theta^2)(\delta^4 z_{11} - \delta^4 z_{1-1} - \delta^4 z_{-11} + \delta^4 z_{-1-1}) \\
&+ \tfrac{1}{480}\theta\chi(1 - \chi^2)(4 - \chi^2)(\delta'^4 z_{11} - \delta'^4 z_{1-1} - \delta'^4 z_{-11} + \delta'^4 z_{-1-1})
\end{aligned}\right] \text{6th order}$$

$$\left.\begin{aligned}
&+ \tfrac{1}{1440}\theta^2\chi(1 - \theta^2)(4 - \theta^2)(\delta^6 z_{01} - \delta^6 z_{0-1}) \\
&+ \tfrac{1}{1440}\theta\chi^2(1 - \chi^2)(4 - \chi^2)(\delta'^6 z_{10} - \delta'^6 z_{-10}) \\
&- \tfrac{1}{10080}\theta(1 - \theta^2)(4 - \theta^2)(9 - \theta^2)(\delta^6 z_{10} - \delta^6 z_{-10}) \\
&- \tfrac{1}{10080}\chi(1 - \chi^2)(4 - \chi^2)(9 - \chi^2)(\delta'^6 z_{01} - \delta'^6 z_{0-1}) \\
&+ \tfrac{1}{288}\theta^2\chi(1 - \theta^2)(1 - \chi^2)(\delta^4\delta'^2 z_{01} - \delta^4\delta'^2 z_{0-1}) \\
&+ \tfrac{1}{288}\theta\chi^2(1 - \theta^2)(1 - \chi^2)(\delta^2\delta'^4 z_{10} - \delta^2\delta'^4 z_{-10}) \\
&+ \tfrac{1}{480}\theta\chi^2(1 - \theta^2)(4 - \theta^2)(\delta^4\delta'^2 z_{10} - \delta^4\delta'^2 z_{-10}) \\
&+ \tfrac{1}{480}\theta^2\chi(1 - \chi^2)(4 - \chi^2)(\delta^2\delta'^4 z_{01} - \delta^2\delta'^4 z_{0-1})
\end{aligned}\right] \text{7th order}$$

$$+ \text{eighth order terms} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(xi)}.$$

The above expression must now, in the first place, be integrated between the limits $-\frac{1}{2}$ and $\frac{1}{2}$ for each of θ and χ. It may be noticed, at the outset, that all the terms with odd order differences vanish on integration, since when integrated the results all contain even powers either of θ or χ or both. Denoting, as before, the portion of the above expression containing rth order differences by Z_r, we have

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} Z_0\, d\theta\, d\chi = z_{00} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(xii)}.$$

* *Tracts for Computers*, III, p. 29.

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} Z_2 d\theta d\chi = \int_{-\frac{1}{2}}^{\frac{1}{2}} d\theta \left[\frac{\chi^3}{6}(1-\tfrac{1}{2}\theta^2)\,\delta'^2 z_{00} + \tfrac{1}{2}\theta^2(\chi - \tfrac{1}{6}\chi^3)\,\delta^2 z_{00} \right.$$

$$\left. + \tfrac{1}{24}\theta^2\chi^3(\delta'^2 z_{10} + \delta'^2 z_{-10} + \delta^2 z_{01} + \delta^2 z_{0-1})\right]_{\chi=-\frac{1}{2}}^{\chi=\frac{1}{2}}$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} d\theta \left[\tfrac{1}{24}(1-\tfrac{1}{2}\theta^2)\,\delta'^2 z_{00} + \tfrac{23}{48}\theta^2\delta^2 z_{00} \right.$$

$$\left. + \tfrac{1}{96}\theta^2(\delta'^2 z_{10} + \delta'^2 z_{-10} + \delta^2 z_{01} + \delta^2 z_{0-1})\right]$$

$$= \tfrac{1}{1152}\left[46(\delta'^2 z_{00} + \delta^2 z_{00}) + \delta'^2 z_{10} + \delta'^2 z_{-10} + \delta^2 z_{01} + \delta^2 z_{0-1}\right] \quad \text{............(xiii).}$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} Z_4 d\theta d\chi = \int_{-\frac{1}{2}}^{\frac{1}{2}} d\theta \left[-\tfrac{1}{24}\theta(1-\theta^2)\frac{\chi^2}{2}(\delta^2 z_{11} - \delta^2 z_{1-1} - \delta^2 z_{-11} + \delta^2 z_{-1-1}) \right.$$

$$- \tfrac{1}{24}\theta\left(\frac{\chi^2}{2} - \frac{\chi^4}{4}\right)(\delta'^2 z_{11} - \delta'^2 z_{1-1} - \delta'^2 z_{-11} + \delta'^2 z_{-1-1})$$

$$\left. - \tfrac{1}{24}\theta^2(1-\theta^2)\,\chi\delta^4 z_{00} - \tfrac{1}{24}\left(\frac{\chi^3}{3} - \frac{\chi^5}{5}\right)\delta'^4 z_{00}\right]_{\chi=-\frac{1}{2}}^{\chi=\frac{1}{2}}$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} d\theta \left[-\tfrac{1}{24}\theta^2(1-\theta^2)\,\delta^4 z_{00} - \tfrac{1}{24}(\tfrac{1}{12} - \tfrac{1}{80})\,\delta'^4 z_{00}\right]$$

$$= -\tfrac{17}{5760}\left[\delta^4 z_{00} + \delta'^4 z_{00}\right] \quad \text{..(xiv).}$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} Z_6 d\theta d\chi = \int_{-\frac{1}{2}}^{\frac{1}{2}} d\theta \left[\tfrac{1}{720}\theta^2(1-\theta^2)(4-\theta^2)\,\chi\delta^6 z_{00} + \tfrac{1}{720}\left(\frac{4\chi^3}{3} - \chi^5 + \frac{\chi^7}{7}\right)\delta'^6 z_{00} \right.$$

$$- \tfrac{1}{48}\theta^2(1-\theta^2)\frac{\chi^3}{3}\,\delta^4\delta'^2 z_{00} - \tfrac{1}{48}\theta^2\left(\frac{\chi^3}{3} - \frac{\chi^5}{5}\right)\delta^2\delta'^4 z_{00}$$

$$+ \tfrac{1}{144}\theta(1-\theta^2)\left(\frac{\chi^2}{2} - \frac{\chi^4}{4}\right)$$

$$\times (\delta^2\delta'^2 z_{11} - \delta^2\delta'^2 z_{1-1} - \delta^2\delta'^2 z_{-11} + \delta^2\delta'^2 z_{-1-1})$$

$$+ \tfrac{1}{480}\theta(1-\theta^2)(4-\theta^2)\frac{\chi^2}{2}(\delta^4 z_{11} - \delta^4 z_{1-1} - \delta^4 z_{-11} + \delta^4 z_{-1-1})$$

$$+ \tfrac{1}{480}\theta\left(2\chi^2 - \frac{5\chi^4}{4} + \frac{\chi^6}{6}\right)$$

$$\left. \times (\delta'^4 z_{11} - \delta'^4 z_{1-1} - \delta'^4 z_{-11} + \delta'^4 z_{-1-1})\right]_{\chi=-\frac{1}{2}}^{\chi=\frac{1}{2}}$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} d\theta \left[\tfrac{1}{720}\theta^2(1-\theta^2)(4-\theta^2)\,\delta^6 z_{00} + \tfrac{1}{720}(\tfrac{1}{3} - \tfrac{1}{16} + \tfrac{1}{448})\,\delta'^6 z_{00} \right.$$

$$\left. - \tfrac{1}{576}\theta^2(1-\theta^2)\,\delta^4\delta'^2 z_{00} - \tfrac{17}{48\times 240}\theta^2\delta^2\delta'^4 z_{00}\right]$$

$$= \tfrac{1}{967680}\left[367(\delta^6 z_{00} + \delta'^6 z_{00}) - 119(\delta^4\delta'^2 z_{00} + \delta^2\delta'^4 z_{00})\right] \quad \text{...(xv).}$$

Combining (xii), (xiii), (xiv), (xv) we have

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} z_{\theta,\chi}\, d\theta\, d\chi = [z_{00} + \tfrac{1}{1152}\{46(\delta'^2 z_{00} + \delta^2 z_{00}) + \delta'^2 z_{10} + \delta'^2 z_{-10} + \delta^2 z_{01} + \delta^2 z_{0-1}\}$$
$$- \tfrac{17}{5760}(\delta^4 z_{00} + \delta'^4 z_{00}) + \tfrac{1}{967680}\{367(\delta^6 z_{00} + \delta'^6 z_{00})$$
$$- 119(\delta^4\delta'^2 z_{00} + \delta^2\delta'^4 z_{00})\}] \text{ ....(xvi).}$$

We now use the relations (given on p. 25) between the operators δ and E to express this result in terms of the z's, and reach the following expressions for $\int_{-\frac{1}{2}h}^{\frac{1}{2}h}\int_{-\frac{1}{2}k}^{\frac{1}{2}k} z\,dx\,dy$:

$$\begin{cases}
(\alpha)\ hkz_{00}, \\
(\beta)\ \dfrac{hk}{576}[484z_{00} + 22(z_{01} + z_{10} + z_{0-1} + z_{-10}) + z_{11} + z_{-1-1} + z_{1-1} + z_{-11}], \\
(\gamma)\ \dfrac{hk}{5760}[4636z_{00} + 288(z_{01} + z_{10} + z_{0-1} + z_{-10}) + 10(z_{11} + z_{-1-1} + z_{1-1} + z_{-11}) \\
\qquad - 17(z_{20} + z_{02} + z_{-20} + z_{0-2})], \\
(\delta)\ \dfrac{hk}{967680}[767024z_{00} + 52223(z_{10} + z_{01} + z_{-10} + z_{0-1}) \\
\qquad + 2632(z_{11} + z_{-1-1} + z_{1-1} + z_{-11}) - 4820(z_{20} + z_{02} + z_{-20} + z_{0-2}) \\
\qquad - 119(z_{21} + z_{12} + z_{-12} + z_{-21} + z_{-2-1} + z_{-1-2} + z_{1-2} + z_{2-1}) \\
\qquad + 367(z_{30} + z_{03} + z_{-30} + z_{0-3})] \text{ ....(xvii).}
\end{cases}$$

These are correct, respectively, up to and including 1st, 3rd, 5th and 7th order differences.

Now by summation over the appropriate panels, we may obtain formulae for $\int_{-\frac{1}{2}h}^{\infty}\int_{-\infty}^{\infty} z\,dx\,dy$, and finally, by a change of origin to the point $-\frac{1}{2}h$, 0, for $\int_0^{\infty}\int_{-\infty}^{\infty} z\,dx\,dy.$

u_r being defined as before as $\sum_{s=-\infty}^{s=\infty} z_{r,s}$ we have the relations:

$$\begin{aligned}
&\Sigma z_{00} &&= u_0 + u_1 + u_2 + \ldots, \\
&\Sigma(z_{01} + z_{10} + z_{0-1} + z_{-10}) &&= u_{-1} + 3u_0 + 4(u_1 + u_2 + u_3 + \ldots), \\
&\Sigma(z_{11} + z_{-1-1} + z_{1-1} + z_{-11}) &&= 2u_{-1} + 2u_0 + 4(u_1 + u_2 + u_3 + \ldots), \\
&\Sigma(z_{20} + z_{02} + z_{-20} + z_{0-2}) &&= u_{-2} + u_{-1} + 3u_0 + 3u_1 + 4(u_2 + u_3 + \ldots), \\
&\Sigma(z_{21} + z_{12} + z_{-12} + z_{-21} + z_{-2-1} + z_{-1-2} + z_{1-2} + z_{2-1}) \\
& &&= 2u_{-2} + 4u_{-1} + 4u_0 + 6u_1 + 8(u_2 + u_3 + \ldots),
\end{aligned}$$

the summation being extended to all the panels between $x = -\frac{1}{2}h$ and ∞, $y = -\infty$ and ∞.

Substituting these results in (xvii) and changing the origin, we obtain, after some reductions, the following expressions for $\int_0^\infty \int_{-\infty}^\infty z\,dx\,dy$:

$$\left\{\begin{array}{ll} (\alpha) & hk\left[u_{\frac{1}{2}} + u_{\frac{3}{2}} + u_{\frac{5}{2}} + \dots\right], \\ (\beta) & \frac{hk}{24}\left[u_{-\frac{1}{2}} + 23u_{\frac{1}{2}} + 24\left(u_{\frac{3}{2}} + u_{\frac{5}{2}} + u_{\frac{7}{2}} + \dots\right)\right], \\ (\gamma) & \frac{hk}{5760}\left[-17u_{-\frac{3}{2}} + 291u_{-\frac{1}{2}} + 5469u_{\frac{1}{2}} + 5777u_{\frac{3}{2}} + 5760\left(u_{\frac{5}{2}} + u_{\frac{7}{2}} + \dots\right)\right], \\ (\delta) & \frac{hk}{967680}\left[367u_{-\frac{5}{2}} - 4691u_{-\frac{3}{2}} + 52558u_{-\frac{1}{2}} + 915122u_{\frac{1}{2}} + 972371u_{\frac{3}{2}}\right. \\ & \qquad \left. + 967313u_{\frac{5}{2}} + 967680\left(u_{\frac{7}{2}} + u_{\frac{9}{2}} + \dots\right)\right] \end{array}\right. \quad \text{.....................(xviii)},$$

the results being correct respectively up to and including 1st, 3rd, 5th and 7th order differences.

It will be noticed immediately, that on putting

$$u_{-\frac{1}{2}} = u_{\frac{1}{2}}, \qquad u_{-\frac{3}{2}} = u_{\frac{3}{2}}, \qquad u_{-\frac{5}{2}} = u_{\frac{5}{2}},$$

for the particular case of a symmetrical surface (xviii) (β), (γ), (δ) all reduce to (xviii) (α). The reason for this has already been explained (p. 28 *et seq.*); in fact the formulae in (xviii) are respectively equivalent to the corresponding formulae in (viii), to the order of approximation to which they go. This may best be seen as follows:

We know that the formulae in (viii) are respectively true up to 1st, 3rd, 5th and 7th order differences; we may therefore suppose

(α) $u_{\frac{1}{2}}$ to lie on the straight line between u_0 and u_1,

(β) $u_{-\frac{1}{2}}$, $u_{\frac{1}{2}}$ and $u_{\frac{3}{2}}$ to lie on the 3rd order parabola through u_{-1}, u_0, u_1 and u_2

(γ) $u_{-\frac{3}{2}}$, $u_{-\frac{1}{2}}$, $u_{\frac{1}{2}}$, $u_{\frac{3}{2}}$, $u_{\frac{5}{2}}$ to lie on the 5th order parabola through

$$u_{-2},\ u_{-1},\ u_0,\ u_1,\ u_2,\ u_3,$$

(δ) $u_{-\frac{5}{2}}$, $u_{-\frac{3}{2}}$, $u_{-\frac{1}{2}}$, $u_{\frac{1}{2}}$, $u_{\frac{3}{2}}$, $u_{\frac{5}{2}}$, $u_{\frac{7}{2}}$ to lie on the 7th order parabola through

$$u_{-3},\ u_{-2},\ u_{-1},\ u_0,\ u_1,\ u_2,\ u_3,\ u_4,$$

and similar results will be true for the higher u's.

The relations between the u's are therefore given by the following Lagrange formulae*:

$$(\alpha) \qquad u_{\frac{1}{2}} = \tfrac{1}{2}(u_0 + u_1) \quad \text{.................................(xix)}.$$

$$(\beta) \quad \left\{\begin{array}{l} u_{-\frac{1}{2}} = \frac{1}{16}\{5u_1 + 15u_0 - 5u_1 + u_2\}, \\ u_{\frac{1}{2}} = \frac{1}{16}\{-u_{-1} + 9u_0 + 9u_1 - u_2\}, \\ u_{\frac{3}{2}} = \frac{1}{16}\{u_{-1} - 5u_0 + 15u_1 + 5u_2\} \end{array}\right. \quad \text{...........(xix)}.$$

* *Tracts for Computers*, II, pp. 31, 44, 45, 46.

$$(\gamma)\left\{\begin{aligned}
u_{-\frac{3}{2}} &= \tfrac{1}{256}\{63u_{-2}+315u_{-1}-210u_0+126u_1-45u_2+7u_3\},\\
u_{-\frac{1}{2}} &= \tfrac{1}{256}\{-7u_{-2}+105u_{-1}+210u_0-70u_1+21u_2-3u_3\},\\
u_{\frac{1}{2}} &= \tfrac{1}{256}\{3u_{-2}-25u_{-1}+150u_0+150u_1-25u_2+3u_3\},\\
u_{\frac{3}{2}} &= \tfrac{1}{256}\{-3u_{-2}+21u_{-1}-70u_0+210u_1+105u_2-7u_3\},\\
u_{\frac{5}{2}} &= \tfrac{1}{256}\{7u_{-2}-45u_{-1}+126u_0-210u_1+315u_2+63u_3\} \text{ ....(xix).}
\end{aligned}\right.$$

$$(\delta)\left\{\begin{aligned}
u_{-\frac{5}{2}} &= \tfrac{1}{2048}\{429u_{-3}+3003u_{-2}-3003u_{-1}+3003u_0-2145u_1+1001u_2\\
&\qquad -273u_3+33u_4\},\\
u_{-\frac{3}{2}} &= \tfrac{1}{2048}\{-33u_{-3}+693u_{-2}+2079u_{-1}-1155u_0+693u_1-297u_2\\
&\qquad +77u_3-9u_4\},\\
u_{-\frac{1}{2}} &= \tfrac{1}{2048}\{9u_{-3}-105u_{-2}+945u_{-1}+1575u_0-525u_1+189u_2-45u_3+5u_4\},\\
u_{\frac{1}{2}} &= \tfrac{1}{2048}\{-5u_{-3}+49u_{-2}-245u_{-1}+1225u_0+1225u_1-245u_2\\
&\qquad +49u_3-5u_4\},\\
u_{\frac{3}{2}} &= \tfrac{1}{2048}\{5u_{-3}-45u_{-2}+189u_{-1}-525u_0+1575u_1+945u_2-105u_3+9u_4\},\\
u_{\frac{5}{2}} &= \tfrac{1}{2048}\{-9u_{-3}+77u_{-2}-297u_{-1}+693u_0-1155u_1+2079u_2\\
&\qquad +693u_3-33u_4\},\\
u_{\frac{7}{2}} &= \tfrac{1}{2048}\{33u_{-3}-273u_{-2}+1001u_{-1}-2145u_0+3003u_1-3003u_2\\
&\qquad +3003u_3+429u_4\} \text{ ....(xix),}
\end{aligned}\right.$$

and similar results hold for the higher u's. Whence we obtain:

$$\left\{\begin{aligned}
&(\alpha)\quad u_{\frac{1}{2}} = \tfrac{1}{2}(u_0+u_1),\\
&(\beta)\quad u_{-\frac{1}{2}}+22u_{\frac{1}{2}}+u_{\frac{3}{2}} = -u_{-1}+13u_0+13u_1-u_2,\\
&(\gamma)\quad \tfrac{1}{5760}(-17u_{-\frac{3}{2}}+308u_{-\frac{1}{2}}+5178u_{\frac{1}{2}}+308u_{\frac{3}{2}}-17u_{\frac{5}{2}})\\
&\qquad = \tfrac{1}{1440}(11u_{-2}-93u_{-1}+802u_0+802u_1-93u_2+11u_3),\\
&(\delta)\quad \tfrac{1}{967680}(367u_{-\frac{5}{2}}-5058u_{-\frac{3}{2}}+57249u_{-\frac{1}{2}}+862564u_{\frac{1}{2}}+57249u_{\frac{3}{2}}\\
&\qquad -5058u_{\frac{5}{2}}+367u_{\frac{7}{2}})\\
&\qquad = \tfrac{1}{120960}(-191u_{-3}+1879u_{-2}-9531u_{-1}+68323u_0+68323u_1\\
&\qquad -9531u_2+1879u_3-191u_4) \text{............(xx),}
\end{aligned}\right.$$

and similar results hold respectively for every set of 1, 3, 5, 7 consecutive u's with fractional suffixes. Therefore summing for every such set we obtain:

$$\left\{\begin{aligned}
&(\alpha)\quad hk[u_{\frac{1}{2}}+u_{\frac{3}{2}}+u_{\frac{5}{2}}+\ldots] = \frac{hk}{2}[u_0+2(u_1+u_2+u_3+\ldots)],\\
&(\beta)\quad \frac{hk}{24}[u_{-\frac{1}{2}}+23u_{\frac{1}{2}}+24(u_{\frac{3}{2}}+u_{\frac{5}{2}}+u_{\frac{7}{2}}+\ldots)] = \frac{hk}{24}[-u_{-1}+12u_0+25u_1\\
&\qquad +24(u_2+u_3+\ldots)],\\
&(\gamma)\quad \frac{hk}{5760}[-17u_{-\frac{3}{2}}+291u_{-\frac{1}{2}}+5469u_{\frac{1}{2}}+5777u_{\frac{3}{2}}+5760(u_{\frac{5}{2}}+u_{\frac{7}{2}}+\ldots)]\\
&\qquad = \frac{hk}{1440}[11u_{-2}-82u_{-1}+720u_0+1522u_1+1429u_2+1440(u_3+u_4+\ldots)],\\
&(\delta)\quad \frac{hk}{967680}[367u_{-\frac{5}{2}}-4691u_{-\frac{3}{2}}+52558u_{-\frac{1}{2}}+915122u_{\frac{1}{2}}+972371u_{\frac{3}{2}}\\
&\qquad +967313u_{\frac{5}{2}}+967680(u_{\frac{7}{2}}+u_{\frac{9}{2}}+\ldots)],\\
&\qquad = \frac{hk}{120960}[-191u_{-3}+1688u_{-2}-7843u_{-1}+60480u_0+128803u_1\\
&\qquad +119272u_2+121151u_3+120960(u_4+u_5+\ldots)] \text{...(xxi),}
\end{aligned}\right.$$

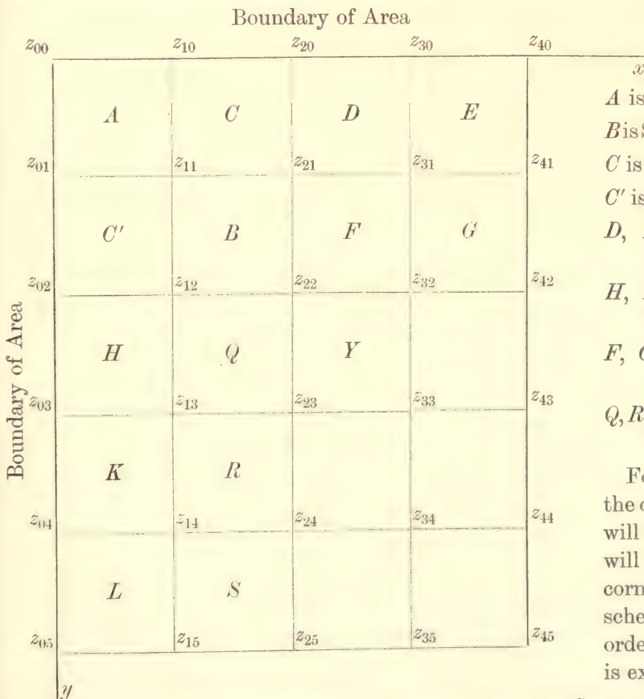
and the identity of formulae (xviii) with formulae (viii) to the given order of approximation is established. Formulae (xviii) may therefore be taken as alternative to formulae (viii) and used if they happen to be more convenient.

It may further be remarked that formulae (xviii), treated as univariate formulae on the u's, are, to the order of differences to which they go, identical with formulae (xvi) of Part I, which must be used instead of them when the ordinates on the negative side of the origin are unknown.

C. Extreme Ordinate Cubature Formulae for $\int_0^{mh}\int_0^{nk} z\,dx\,dy$.

We now turn to the more general problem of finding cubature formulae for the integral $\int_0^{mh}\int_0^{nk} z\,dx\,dy$. These formulae, when established, will clearly be applicable to any double integral where the limits of integration are constants, provided the function z has continuous curvature.

The difficulty, as we have already pointed out, lies in the exclusion of those ordinates which are outside the boundaries of the area over which the function is to be integrated. This difficulty can be met, however, if, in the integration over the panels which lie on the boundary of the rectangle $x = 0$ to mh, $y = 0$ to nk, we use bivariate central difference boundary formulae*. We show the scheme below:



A is First Panel-First Panel.
B is Second Panel-Second Panel.
C is First Panel-Second Panel.
C' is Second Panel-First Panel.
D, E, etc. are First Panel-Ordinary.
H, K, L, etc. are Ordinary-First Panel.
F, G, etc. are Second Panel-Ordinary.
Q, R, S, etc. are Ordinary-Second Panel.

For the remaining meshes the ordinary mid-panel formula will be used. Similar formulae will be used at the other three corners. The modification this scheme undergoes, when sixth order differences are retained, is explained on p. 38.

* These formulae are all given in *Tracts for Computers*, III, p. 49.

We will now integrate, successively, the necessary boundary interpolation formulae. These are given correct up to and including 6th order differences, but when we come to combine the formulae for the separate panels to obtain general formulae for $\int_0^{mh}\int_0^{nk} z\,dx\,dy$, we shall not take into account 6th order terms, as the general 6th order formulae would involve too many ordinates to be of practical use.

In Table IV, however, we give 6th order formulae for the separate panels. If these are used, the computer will require third panel formulae at the boundaries; for panel D a 'First Panel-Third Panel' formula, for F a 'Second Panel-Third Panel' formula, for Y a 'Third Panel-Third Panel' and for H, Q 'Third Panel-First Panel' and 'Third Panel-Second Panel' formulae respectively; while in the remaining panels of the third row and column he will require respectively 'Third Panel-Ordinary' and 'Ordinary-Third Panel' formulae. There will be a similar arrangement at each corner of the rectangle of integration. The necessary formulae are given in Table IV.

We now proceed to the integration.

First Panel-First Panel.

The formula is:

$$\begin{aligned}
z_{\theta,\chi} = {} & (1-\theta)(1-\chi)z_{00} + (1-\chi)\theta z_{10} + (1-\theta)\chi z_{01} + \theta\chi z_{11} \\
& - \tfrac{1}{6}\chi(1-\chi)\left[(5-\chi)\{(1-\theta)\delta'^2 z_{01} + \theta\delta'^2 z_{11}\} - (2-\chi)\{(1-\theta)\delta'^2 z_{02} + \theta\delta'^2 z_{12}\}\right] \\
& - \tfrac{1}{6}\theta(1-\theta)\left[(5-\theta)\{(1-\chi)\delta^2 z_{10} + \chi\delta^2 z_{11}\} - (2-\theta)\{(1-\chi)\delta^2 z_{20} + \chi\delta^2 z_{21}\}\right] \\
& - \tfrac{1}{120}\theta(1-\theta)(2-\theta)(3-\theta)\left[(9-\theta)\{(1-\chi)\delta^4 z_{20} + \chi\delta^4 z_{21}\}\right. \\
& \qquad\qquad \left. - (4-\theta)\{(1-\chi)\delta^4 z_{30} + \chi\delta^4 z_{31}\}\right] \\
& - \tfrac{1}{120}\chi(1-\chi)(2-\chi)(3-\chi)\left[(9-\chi)\{(1-\theta)\delta'^4 z_{02} + \theta\delta'^4 z_{12}\}\right. \\
& \qquad\qquad \left. - (4-\chi)\{(1-\theta)\delta'^4 z_{03} + \theta\delta'^4 z_{13}\}\right] \\
& + \tfrac{1}{36}\theta(1-\theta)\chi(1-\chi)\left[(5-\chi)\{(5-\theta)\delta^2\delta'^2 z_{11} - (2-\theta)\delta^2\delta'^2 z_{21}\}\right. \\
& \qquad\qquad \left. - (2-\chi)\{(5-\theta)\delta^2\delta'^2 z_{12} - (2-\theta)\delta^2\delta'^2 z_{22}\}\right] \\
& - \tfrac{1}{720}\theta(1-\theta)(2-\theta)(3-\theta)(4-\theta)(5-\theta)\left[(1-\chi)\delta^6 z_{30} + \chi\delta^6 z_{31}\right] \\
& - \tfrac{1}{720}\chi(1-\chi)(2-\chi)(3-\chi)(4-\chi)(5-\chi)\left[(1-\theta)\delta'^6 z_{03} + \theta\delta'^6 z_{13}\right] \\
& + \tfrac{1}{48}\theta(1-\theta)\chi(1-\chi)\left[(2-\theta)(3-\theta)\delta^4\delta'^2 z_{21} + (2-\chi)(3-\chi)\delta^2\delta'^4 z_{12}\right]
\end{aligned}$$

............(xxii).

Denoting, as usual, the terms in the above formula containing rth order differences by Z_r we have:

$$\int_0^1\int_0^1 Z_0 d\theta d\chi = \tfrac{1}{4}[z_{00}+z_{10}+z_{01}+z_{11}] \text{ ....(xxiii).}$$

$$\begin{aligned}\int_0^1\int_0^1 Z_2 d\theta d\chi = \int_0^1 d\theta\, [&-\tfrac{1}{8}\{(1-\theta)\,\delta'^2 z_{01}+\theta\delta'^2 z_{11}\}+\tfrac{1}{24}\{(1-\theta)\,\delta'^2 z_{02}+\theta\delta'^2 z_{12}\}\\ &-\tfrac{1}{6}\theta(1-\theta)(5-\theta)\,\tfrac{1}{2}(\delta^2 z_{10}+\delta^2 z_{11})\\ &+\tfrac{1}{6}\theta(1-\theta)(2-\theta)\,\tfrac{1}{2}(\delta^2 z_{20}+\delta^2 z_{21})]\\ = -\tfrac{1}{16}(\delta'^2 z_{01}+\delta'^2 z_{11}&+\delta^2 z_{10}+\delta^2 z_{11})+\tfrac{1}{48}(\delta'^2 z_{02}+\delta'^2 z_{12}+\delta^2 z_{20}+\delta^2 z_{21})\\ &\text{...........(xxiv).}\end{aligned}$$

$$\begin{aligned}\int_0^1\int_0^1 Z_4 d\theta d\chi = \int_0^1 d\theta\, [&-\tfrac{1}{120}\theta(1-\theta)(2-\theta)(3-\theta)(9-\theta)\,\tfrac{1}{2}(\delta^4 z_{20}+\delta^4 z_{21})\\ &+\tfrac{1}{120}\theta(1-\theta)(2-\theta)(3-\theta)(4-\theta)\,\tfrac{1}{2}(\delta^4 z_{30}+\delta^4 z_{31})\\ &-\tfrac{65}{1440}\{(1-\theta)\,\delta'^4 z_{02}+\theta\delta'^4 z_{12}\}+\tfrac{3}{160}\{(1-\theta)\,\delta'^4 z_{03}+\theta\delta'^4 z_{13}\}\\ &+\tfrac{1}{36}\theta(1-\theta)(5-\theta)\,\tfrac{3}{4}\delta^2\delta'^2 z_{11}-\tfrac{1}{36}\theta(1-\theta)(2-\theta)\,\tfrac{3}{4}\delta^2\delta'^2 z_{21}\\ &-\tfrac{1}{36}\theta(1-\theta)(5-\theta)\,\tfrac{1}{4}\delta^2\delta'^2 z_{12}+\tfrac{1}{36}\theta(1-\theta)(2-\theta)\,\tfrac{1}{4}\delta^2\delta'^2 z_{22}]\\ = \tfrac{1}{2880}[-65(\delta^4 z_{20}+\delta^4 z_{21}&+\delta'^4 z_{02}+\delta'^4 z_{12})+27(\delta^4 z_{30}+\delta^4 z_{31}+\delta'^4 z_{03}+\delta'^4 z_{13})\\ &+45\,\delta^2\delta'^2 z_{11}-15(\delta^2\delta'^2 z_{21}+\delta^2\delta'^2 z_{12})+5\,\delta^2\delta'^2 z_{22}] \text{ ...(xxv).}\end{aligned}$$

$$\begin{aligned}\int_0^1\int_0^1 Z_6 d\theta d\chi = \int_0^1 d\theta\, [&-\tfrac{1}{720}\theta(1-\theta)(2-\theta)(3-\theta)(4-\theta)(5-\theta)\,\tfrac{1}{2}(\delta^6 z_{30}+\delta^6 z_{31})\\ &-\tfrac{1}{720}\cdot\tfrac{863}{84}\,\tfrac{1}{2}(\delta'^6 z_{03}+\delta'^6 z_{13})\\ &+\tfrac{1}{48}\theta(1-\theta)(2-\theta)(3-\theta)\,\tfrac{1}{6}\delta^4\delta'^2 z_{21}+\tfrac{1}{48}\theta(1-\theta)\,\tfrac{19}{30}\delta^2\delta'^4 z_{12}]\\ = \tfrac{1}{120960}[-863\,(\delta^6 z_{30}&+\delta^6 z_{31}+\delta'^6 z_{03}+\delta'^6 z_{13})\\ &+266\,(\delta^4\delta'^2 z_{21}+\delta^2\delta'^4 z_{12})] \text{...(xxvi).}\end{aligned}$$

Combining (xxiii), (xxiv), (xxv) and (xxvi) we reach

$$\begin{aligned}\int_0^1\int_0^1 z_{\theta,\chi}\, d\theta d\chi = [&\tfrac{1}{4}(z_{00}+z_{10}+z_{01}+z_{11})-\tfrac{1}{16}(\delta'^2 z_{01}+\delta'^2 z_{11}+\delta^2 z_{10}+\delta^2 z_{11})\\ &+\tfrac{1}{48}(\delta'^2 z_{02}+\delta'^2 z_{12}+\delta^2 z_{20}+\delta^2 z_{21})\\ &+\tfrac{1}{2880}\{-65\,(\delta^4 z_{20}+\delta^4 z_{21}+\delta'^4 z_{02}+\delta'^4 z_{12})\\ &\qquad+27\,(\delta^4 z_{30}+\delta^4 z_{31}+\delta'^4 z_{03}+\delta'^4 z_{13})\\ &\qquad+45\,\delta^2\delta'^2 z_{11}-15\,(\delta^2\delta'^2 z_{21}+\delta^2\delta'^2 z_{12})+5\,\delta^2\delta'^2 z_{22}\}\\ &+\tfrac{1}{120960}\{-863\,(\delta^6 z_{30}+\delta^6 z_{31}+\delta'^6 z_{03}+\delta'^6 z_{13})\\ &\qquad+266\,(\delta^4\delta'^2 z_{21}+\delta^2\delta'^4 z_{12})\}] \text{ ....(xxvii).}\end{aligned}$$

If we now express, as usual, the differences in terms of the z's, we obtain the following three expressions for $\int_0^h \int_0^k z\,dx\,dy$, respectively correct up to and including 1st, 3rd and 5th order differences*:

$$\left\{\begin{array}{ll}
(\alpha) & \dfrac{hk}{4}(z_{00}+z_{10}+z_{01}+z_{11}), \\[2ex]
(\beta) & \dfrac{hk}{48}\begin{bmatrix} 6z_{00}+16z_{10}-5z_{20}+z_{30} \\ +16z_{01}+26z_{11}-5z_{21}+z_{31} \\ -\ 5z_{02}-\ 5z_{12} \\ +\ z_{03}+\ z_{13} \end{bmatrix}, \\[2ex]
(\gamma) & \dfrac{hk}{2880}\begin{bmatrix} 275z_{00}+1077z_{10}-723z_{20}+467z_{30}-173z_{40}+27z_{50} \\ +1077z_{01}+2379z_{11}-973z_{21}+517z_{31}-173z_{41}+27z_{51} \\ -\ 723z_{02}-\ 973z_{12}+125z_{22}-\ 25z_{32} \\ +\ 467z_{03}+\ 517z_{13}-\ 25z_{23}+\ 5z_{33} \\ -\ 173z_{04}-\ 173z_{14} \\ +\ 27z_{05}+\ 27z_{15} \end{bmatrix}
\end{array}\right. \qquad \text{(xxviii)}.$$

First Panel-Second Panel.

The formula is:

$$\begin{aligned}
z_{\theta,\chi} = {} & (1-\theta)(1-\chi)z_{10}+(1-\theta)\chi z_{11}+\theta(1-\chi)z_{20}+\theta\chi z_{21} \\
& -\tfrac{1}{6}\chi(1-\chi)[(5-\chi)\{(1-\theta)\delta'^2 z_{11}+\theta\delta'^2 z_{21}\}-(2-\chi)\{(1-\theta)\delta'^2 z_{12}+\theta\delta'^2 z_{22}\}] \\
& -\tfrac{1}{6}\theta(1-\theta)[(2-\theta)\{(1-\chi)\delta^2 z_{10}+\chi\delta^2 z_{11}\}+(1+\theta)\{(1-\chi)\delta^2 z_{20}+\chi\delta^2 z_{21}\}] \\
& -\tfrac{1}{120}\chi(1-\chi)(2-\chi)(3-\chi)[(9-\chi)\{(1-\theta)\delta'^4 z_{12}+\theta\delta'^4 z_{22}\} \\
& \qquad\qquad -(4-\chi)\{(1-\theta)\delta'^4 z_{13}+\theta\delta'^4 z_{23}\}] \\
& +\tfrac{1}{120}\theta(1-\theta)(1+\theta)(2-\theta)[(8-\theta)\{(1-\chi)\delta^4 z_{20}+\chi\delta^4 z_{21}\} \\
& \qquad\qquad -(3-\theta)\{(1-\chi)\delta^4 z_{30}+\chi\delta^4 z_{31}\}] \\
& +\tfrac{1}{36}\theta(1-\theta)\chi(1-\chi)[(5-\chi)\{(2-\theta)\delta^2\delta'^2 z_{11}+(1+\theta)\delta^2\delta'^2 z_{21}\} \\
& \qquad\qquad -(2-\chi)\{(2-\theta)\delta^2\delta'^2 z_{12}+(1+\theta)\delta^2\delta'^2 z_{22}\}] \\
& -\tfrac{1}{720}\chi(1-\chi)(2-\chi)(3-\chi)(4-\chi)(5-\chi)[(1-\theta)\delta'^6 z_{13}+\theta\delta'^6 z_{23}] \\
& +\tfrac{1}{720}\theta(1-\theta)(1+\theta)(2-\theta)(3-\theta)(4-\theta)[(1-\chi)\delta^6 z_{30}+\chi\delta^6 z_{31}] \\
& +\tfrac{1}{144}\theta(1-\theta)\chi(1-\chi)(2-\chi)(3-\chi)[(2-\theta)\delta^2\delta'^4 z_{12}+(1+\theta)\delta^2\delta'^4 z_{22}] \\
& -\tfrac{1}{48}\theta(1-\theta)\chi(1-\chi)(1+\theta)(2-\theta)\delta^4\delta'^2 z_{21} \quad \ldots\ldots\ldots\ldots\text{(xxix)}.
\end{aligned}$$

* In this and the succeeding panel formulae, cubature formulae in terms of z's up to and including 6th order differences are not given in the text, but they will be found in Table IV.

With the usual notation, we have:

$$\int_0^1\int_0^1 Z_0 d\theta d\chi = \tfrac{1}{4}(z_{11}+z_{10}+z_{20}+z_{21}) \quad \ldots\ldots\ldots\ldots\ldots(\text{xxx}).$$

$$\begin{aligned}\int_0^1\int_0^1 Z_2 d\theta d\chi = \int_0^1 d\theta\,[&-\tfrac{1}{8}\{(1-\theta)\,\delta'^2 z_{11}+\theta\delta'^2 z_{21}\}+\tfrac{1}{24}\{(1-\theta)\,\delta'^2 z_{12}+\theta\delta'^2 z_{22}\}\\ &-\tfrac{1}{6}\theta(1-\theta)(2-\theta)\,\tfrac{1}{2}(\delta^2 z_{10}+\delta^2 z_{11})\\ &-\tfrac{1}{6}\theta(1-\theta^2)\,\tfrac{1}{2}(\delta^2 z_{20}+\delta^2 z_{21})]\\ = -\tfrac{1}{16}(\delta'^2 z_{11}&+\delta'^2 z_{21})+\tfrac{1}{48}(\delta'^2 z_{12}+\delta'^2 z_{22})\\ &-\tfrac{1}{48}(\delta^2 z_{10}+\delta^2 z_{11}+\delta^2 z_{20}+\delta^2 z_{21}) \ldots\ldots\ldots\ldots\ldots\ldots(\text{xxxi}).\end{aligned}$$

$$\begin{aligned}\int_0^1\int_0^1 Z_4 d\theta d\chi = \int_0^1 d\theta\,[&-\tfrac{1}{120}\cdot\tfrac{65}{12}\{(1-\theta)\,\delta'^4 z_{12}+\theta\delta'^4 z_{22}\}\\ &+\tfrac{1}{120}\cdot\tfrac{9}{4}\{(1-\theta)\,\delta'^4 z_{13}+\theta\delta'^4 z_{23}\}\\ &+\tfrac{1}{120}\theta(1-\theta^2)(2-\theta)(8-\theta)\,\tfrac{1}{2}(\delta^4 z_{20}+\delta^4 z_{21})\\ &-\tfrac{1}{120}\theta(1-\theta^2)(2-\theta)(3-\theta)\,\tfrac{1}{2}(\delta^4 z_{30}+\delta^4 z_{31})\\ &+\tfrac{1}{36}\theta(1-\theta)(2-\theta)(\tfrac{3}{4}\delta^2\delta'^2 z_{11})+\tfrac{1}{36}\theta(1-\theta^2)(\tfrac{3}{4}\delta^2\delta'^2 z_{21})\\ &-\tfrac{1}{36}\theta(1-\theta)(2-\theta)(\tfrac{1}{4}\delta^2\delta'^2 z_{12})-\tfrac{1}{36}\theta(1-\theta^2)(\tfrac{1}{4}\delta^2\delta'^2 z_{22})]\\ = \tfrac{1}{2880}[-65(\delta'^4 z_{12}&+\delta'^4 z_{22})+27(\delta'^4 z_{13}+\delta'^4 z_{23})\\ &+33(\delta^4 z_{20}+\delta^4 z_{21})-11(\delta^4 z_{30}+\delta^4 z_{31})\\ &+15(\delta^2\delta'^2 z_{11}+\delta^2\delta'^2 z_{21})-5(\delta^2\delta'^2 z_{12}+\delta^2\delta'^2 z_{22})] \ldots(\text{xxxii}).\end{aligned}$$

$$\begin{aligned}\int_0^1\int_0^1 Z_6 d\theta d\chi = \int_0^1 d\theta\,[&-\tfrac{1}{720}\cdot\tfrac{863}{84}\{(1-\theta)\,\delta'^6 z_{13}+\theta\delta'^6 z_{23}\}\\ &+\tfrac{1}{720}\theta(1-\theta)(2-\theta)(3-\theta)(4-\theta)[\tfrac{1}{2}\delta^6 z_{30}+\tfrac{1}{2}\delta^6 z_{31}]\\ &+\tfrac{1}{144}\theta(1-\theta)(2-\theta)(\tfrac{19}{30}\delta^2\delta'^4 z_{12})+\tfrac{1}{144}\theta(1-\theta^2)\,\tfrac{19}{30}\delta^2\delta'^4 z_{22}\\ &-\tfrac{1}{48}\theta(1-\theta^2)(2-\theta)\,\tfrac{1}{6}\delta^4\delta'^2 z_{21}]\\ = \tfrac{1}{120960}[-863(\delta'^6 z_{13}&+\delta'^6 z_{23})+271(\delta^6 z_{30}+\delta^6 z_{31})\\ &+133(\delta^2\delta'^4 z_{12}+\delta^2\delta'^4 z_{22})-154\,\delta^4\delta'^2 z_{21}] \ldots\ldots\ldots\ldots(\text{xxxiii}).\end{aligned}$$

Combining (xxx), (xxxi), (xxxii) and (xxxiii) we have

$$\begin{aligned}\int_0^1\int_0^1 z_{\theta,\chi} d\theta d\chi = [&\tfrac{1}{4}(z_{11}+z_{10}+z_{20}+z_{21})+\tfrac{1}{48}\{\delta'^2 z_{12}+\delta'^2 z_{22}-3(\delta'^2 z_{11}+\delta'^2 z_{21})\\ &\qquad-(\delta^2 z_{10}+\delta^2 z_{11}+\delta^2 z_{20}+\delta^2 z_{21})\}\\ &+\tfrac{1}{2880}\{-65(\delta'^4 z_{12}+\delta'^4 z_{22})+27(\delta'^4 z_{13}+\delta'^4 z_{23})\\ &\qquad+33(\delta^4 z_{20}+\delta^4 z_{21})-11(\delta^4 z_{30}+\delta^4 z_{31})\\ &\qquad+15(\delta^2\delta'^2 z_{11}+\delta^2\delta'^2 z_{21})-5(\delta^2\delta'^2 z_{12}+\delta^2\delta'^2 z_{22})\}\\ &+\tfrac{1}{120960}\{-863(\delta'^6 z_{13}+\delta'^6 z_{23})+271(\delta^6 z_{30}+\delta^6 z_{31})\\ &\qquad+133(\delta^2\delta'^4 z_{12}+\delta^2\delta'^4 z_{22})-154\,\delta^4\delta'^2 z_{21}\}] \ldots(\text{xxxiv}).\end{aligned}$$

Expressing the differences in terms of the z's, we have the following expressions for $\int_h^{2h}\int_0^k z\,dx\,dy$:

$$\left\{\begin{array}{ll}(\alpha) & \dfrac{hk}{4}(z_{11}+z_{10}+z_{20}+z_{21}),\\[2ex] (\beta) & \dfrac{hk}{48}\left[\begin{array}{l}-z_{00}+10z_{10}+10z_{20}-z_{30}\\ -z_{01}+20z_{11}+20z_{21}-z_{31}\\ \qquad -\ 5z_{12}-\ 5z_{22}\\ \qquad +\ \ z_{13}+\ \ z_{23}\end{array}\right],\\[2ex] (\gamma) & \dfrac{hk}{2880}\left[\begin{array}{l}-12z_{00}+\ 377z_{10}+\ 762z_{20}-243z_{30}+77z_{40}-11z_{50}\\ -62z_{01}+1379z_{11}+1764z_{21}-293z_{31}+77z_{41}-11z_{51}\\ +25z_{02}-\ 823z_{12}-\ 823z_{22}+\ 25z_{32}\\ -\ 5z_{03}+\ 487z_{13}+\ 487z_{23}-\ \ 5z_{33}\\ \qquad -\ 173z_{14}-\ 173z_{24}\\ \qquad +\ \ 27z_{15}+\ \ 27z_{25}\end{array}\right]\end{array}\right. \quad \ldots\text{(xxxv)}.$$

These are correct, respectively, up to and including 1st, 3rd and 5th order differences.

Second Panel-Second Panel.

The formula is:

$$\begin{aligned}z_{\theta,\chi} = {} & (1-\theta)\chi z_{12} + (1-\theta)(1-\chi)z_{11} + \theta(1-\chi)z_{21} + \theta\chi z_{22}\\ & -\tfrac{1}{6}\chi(1-\chi)[(2-\chi)\{(1-\theta)\delta'^2 z_{11} + \theta\delta'^2 z_{21}\} + (1+\chi)\{(1-\theta)\delta'^2 z_{12} + \theta\delta'^2 z_{22}\}]\\ & -\tfrac{1}{6}\theta(1-\theta)[(2-\theta)\{(1-\chi)\delta^2 z_{11} + \chi\delta^2 z_{12}\} + (1+\theta)\{(1-\chi)\delta^2 z_{21} + \chi\delta^2 z_{22}\}]\\ & +\tfrac{1}{120}\chi(1-\chi)(1+\chi)(2-\chi)[(8-\chi)\{(1-\theta)\delta'^4 z_{12} + \theta\delta'^4 z_{22}\}\\ & \qquad\qquad -(3-\chi)\{(1-\theta)\delta'^4 z_{13} + \theta\delta'^4 z_{23}\}]\\ & +\tfrac{1}{120}\theta(1-\theta)(1+\theta)(2-\theta)[(8-\theta)\{(1-\chi)\delta^4 z_{21} + \chi\delta^4 z_{22}\}\\ & \qquad\qquad -(3-\theta)\{(1-\chi)\delta^4 z_{31} + \chi\delta^4 z_{32}\}]\\ & +\tfrac{1}{36}\theta(1-\theta)\chi(1-\chi)[(2-\chi)\{(2-\theta)\delta^2\delta'^2 z_{11} + (1+\theta)\delta^2\delta'^2 z_{21}\}\\ & \qquad\qquad +(1+\chi)\{(2-\theta)\delta^2\delta'^2 z_{12} + (1+\theta)\delta^2\delta'^2 z_{22}\}]\\ & +\tfrac{1}{720}\chi(1-\chi)(2-\chi)(1+\chi)(3-\chi)(4-\chi)[(1-\theta)\delta'^6 z_{13} + \theta\delta'^6 z_{23}]\\ & +\tfrac{1}{720}\theta(1-\theta)(1+\theta)(2-\theta)(3-\theta)(4-\theta)[(1-\chi)\delta^6 z_{31} + \chi\delta^6 z_{32}]\\ & -\tfrac{1}{144}\theta(1-\theta)\chi(1-\chi)(1+\chi)(2-\chi)[(2-\theta)\delta^2\delta'^4 z_{12} + (1+\theta)\delta^2\delta'^4 z_{22}]\\ & -\tfrac{1}{144}\theta(1-\theta)(1+\theta)(2-\theta)(1-\chi)[(2-\chi)\delta^4\delta'^2 z_{21} + (1+\chi)\delta^4\delta'^2 z_{22}]\end{aligned}$$

............(xxxvi).

We have $$\int_0^1\int_0^1 Z_0 d\theta d\chi = \tfrac{1}{4}(z_{12}+z_{11}+z_{21}+z_{22}) \quad\text{............(xxxvii).}$$

$$\begin{aligned}\int_0^1\int_0^1 Z_2 d\theta d\chi = \int_0^1 d\theta\, [&-\tfrac{1}{6}\cdot\tfrac{1}{4}\{(1-\theta)\,\delta'^2 z_{11}+\theta\delta'^2 z_{21}\} \\ &-\tfrac{1}{6}\cdot\tfrac{1}{4}\{(1-\theta)\,\delta'^2 z_{12}+\theta\delta'^2 z_{22}\} \\ &-\tfrac{1}{6}\theta(1-\theta)(2-\theta)\tfrac{1}{2}(\delta^2 z_{11}+\delta^2 z_{12}) \\ &-\tfrac{1}{6}\theta(1-\theta^2)\tfrac{1}{2}(\delta^2 z_{21}+\delta^2 z_{22})] \\ = -\tfrac{1}{48}[\delta^2 z_{11}+\delta^2 z_{12}+\delta^2 z_{21}+\delta^2 z_{22}&+\delta'^2 z_{11}+\delta'^2 z_{12}+\delta'^2 z_{21}+\delta'^2 z_{22}] \\ &\text{............(xxxviii).}\end{aligned}$$

$$\begin{aligned}\int_0^1\int_0^1 Z_4 d\theta d\chi = \int_0^1 d\theta\, [&\tfrac{1}{120}\cdot\tfrac{11}{4}\{(1-\theta)\,\delta'^4 z_{12}+\theta\delta'^4 z_{22}\}-\tfrac{1}{120}\cdot\tfrac{11}{12}\{(1-\theta)\,\delta'^4 z_{13}+\theta\delta'^4 z_{23}\} \\ &+\tfrac{1}{120}\theta(1-\theta^2)(2-\theta)(8-\theta)\tfrac{1}{2}(\delta^4 z_{21}+\delta^4 z_{22}) \\ &-\tfrac{1}{120}\theta(1-\theta^2)(2-\theta)(3-\theta)\tfrac{1}{2}(\delta^4 z_{31}+\delta^4 z_{32}) \\ &+\tfrac{1}{36}\theta(1-\theta)(2-\theta)\tfrac{1}{4}\delta^2\delta'^2 z_{11}+\tfrac{1}{36}\theta(1-\theta^2)\tfrac{1}{4}\delta^2\delta'^2 z_{21} \\ &+\tfrac{1}{36}\theta(1-\theta)(2-\theta)\tfrac{1}{4}\delta^2\delta'^2 z_{12}+\tfrac{1}{36}\theta(1-\theta^2)\tfrac{1}{4}\delta^2\delta'^2 z_{22}] \\ = [\tfrac{11}{960}(\delta'^4 z_{12}+\delta'^4 z_{22}+\delta^4 z_{21}+\delta^4 z_{22})&-\tfrac{11}{2880}(\delta'^4 z_{13}+\delta'^4 z_{23}+\delta^4 z_{31}+\delta^4 z_{32}) \\ &+\tfrac{1}{576}(\delta^2\delta'^2 z_{11}+\delta^2\delta'^2 z_{21}+\delta^2\delta'^2 z_{12}+\delta^2\delta'^2 z_{22})] \quad\text{...(xxxix).}\end{aligned}$$

$$\begin{aligned}\int_0^1\int_0^1 Z_6 d\theta d\chi = \int_0^1 d\theta\, [&\tfrac{1}{720}\cdot\tfrac{271}{84}\{(1-\theta)\,\delta'^6 z_{13}+\theta\delta'^6 z_{23}\} \\ &+\tfrac{1}{720}\theta(1-\theta^2)(2-\theta)(3-\theta)(4-\theta)\tfrac{1}{2}(\delta^6 z_{31}+\delta^6 z_{32}) \\ &-\tfrac{1}{144}\cdot\tfrac{11}{30}\theta(1-\theta)(2-\theta)\,\delta^2\delta'^4 z_{12}-\tfrac{1}{144}\cdot\tfrac{11}{30}\theta(1-\theta^2)\,\delta^2\delta'^4 z_{22} \\ &-\tfrac{1}{144}\theta(1-\theta^2)(2-\theta)\tfrac{1}{4}(\delta^4\delta'^2 z_{12}+\delta^4\delta'^2 z_{22})] \\ = [\tfrac{271}{120960}(\delta'^6 z_{13}+\delta'^6 z_{23}&+\delta^6 z_{31}+\delta^6 z_{32}) \\ &-\tfrac{11}{17280}(\delta^2\delta'^4 z_{12}+\delta^2\delta'^4 z_{22}+\delta^4\delta'^2 z_{21}+\delta^4\delta'^2 z_{22})] \quad\text{......(xl).}\end{aligned}$$

Combining (xxxvii), (xxxviii), (xxxix) and (xl) we have

$$\begin{aligned}\int_0^1\int_0^1 z_{\theta,\chi}\, d\theta d\chi = [&\tfrac{1}{4}(z_{12}+z_{11}+z_{21}+z_{22}) \\ &-\tfrac{1}{48}(\delta^2 z_{11}+\delta^2 z_{12}+\delta^2 z_{21}+\delta^2 z_{22}+\delta'^2 z_{11}+\delta'^2 z_{12}+\delta'^2 z_{21}+\delta'^2 z_{22}) \\ &+\tfrac{11}{960}\begin{pmatrix}\delta'^4 z_{12}+\delta'^4 z_{22}\\ +\delta^4 z_{21}+\delta^4 z_{22}\end{pmatrix}-\tfrac{11}{2880}\begin{pmatrix}\delta'^4 z_{13}+\delta'^4 z_{23}\\ +\delta^4 z_{31}+\delta^4 z_{32}\end{pmatrix} \\ &+\tfrac{1}{576}(\delta^2\delta'^2 z_{11}+\delta^2\delta'^2 z_{12}+\delta^2\delta'^2 z_{21}+\delta^2\delta'^2 z_{22}) \\ &+\tfrac{1}{120960}\{271(\delta'^6 z_{13}+\delta'^6 z_{23}+\delta^6 z_{31}+\delta^6 z_{32}) \\ &\qquad -77(\delta^2\delta'^4 z_{12}+\delta^2\delta'^4 z_{22}+\delta^4\delta'^2 z_{21}+\delta^4\delta'^2 z_{22})\}] \quad\text{...(xli).}\end{aligned}$$

Expressing the differences in terms of the z's, we have the following three expressions for $\int_h^{2h}\int_k^{2k} z\,dx\,dy$:

$$\left\{\begin{array}{ll}
(\alpha) & \dfrac{hk}{4}(z_{12}+z_{11}+z_{21}+z_{22}), \\[2ex]
(\beta) & \dfrac{hk}{48}\left[\begin{array}{l} \quad - \ z_{01} - \ z_{02} \\ - z_{10} + 14z_{11} + 14z_{12} - z_{13} \\ - z_{20} + 14z_{21} + 14z_{22} - z_{23} \\ \quad - \ z_{31} + \ z_{32} \end{array}\right], \\[6ex]
(\gamma) & \dfrac{hk}{2880}\left[\begin{array}{l} \quad 5z_{00} - 32z_{01} - 32z_{02} + 5z_{03} \\ - 32z_{10} + 559z_{11} + 944z_{12} - 263z_{13} + 77z_{14} - 11z_{15} \\ - 32z_{20} + 944z_{21} + 1329z_{22} - 263z_{23} + 77z_{24} - 11z_{25} \\ + \ 5z_{30} - 263z_{31} - 263z_{32} + 5z_{33} \\ \qquad + 77z_{41} + 77z_{42} \\ \qquad - 11z_{51} - 11z_{52} \end{array}\right]
\end{array}\right. \quad \ldots\text{(xlii)}.$$

These are correct, respectively, up to and including 1st, 3rd and 5th order differences.

First Panel-Ordinary.

The formula is:

$$\begin{aligned}
z_{\theta,\chi} = {} & (1-\theta)(1-\chi)z_{00} + (1-\theta)\chi z_{01} + \theta(1-\chi)z_{10} + \theta\chi z_{11} \\
& - \tfrac{1}{6}\theta(1-\theta)[(2-\theta)\{(1-\chi)\delta^2 z_{00} + \chi\delta^2 z_{01}\} + (1+\theta)\{(1-\chi)\delta^2 z_{10} + \chi\delta^2 z_{11}\}] \\
& - \tfrac{1}{6}\chi(1-\chi)[(5-\chi)\{(1-\theta)\delta'^2 z_{01} + \theta\delta'^2 z_{11}\} - (2-\chi)\{(1-\theta)\delta'^2 z_{02} + \theta\delta'^2 z_{12}\}] \\
& + \tfrac{1}{120}\theta(1-\theta)(1+\theta)(2-\theta)[(3-\theta)\{(1-\chi)\delta^4 z_{00} + \chi\delta^4 z_{01}\} \\
& \qquad + (2+\theta)\{(1-\chi)\delta^4 z_{10} + \chi\delta^4 z_{11}\}] \\
& + \tfrac{1}{120}\chi(1-\chi)(2-\chi)(3-\chi)[(4-\chi)\{(1-\theta)\delta'^4 z_{03} + \theta\delta'^4 z_{13}\} \\
& \qquad - (9-\chi)\{(1-\theta)\delta'^4 z_{02} + \theta\delta'^4 z_{12}\}] \\
& + \tfrac{1}{36}\theta(1-\theta)\chi(1-\chi)[(5-\chi)\{(2-\theta)\delta^2\delta'^2 z_{01} + (1+\theta)\delta^2\delta'^2 z_{11}\} \\
& \qquad - (2-\chi)\{(2-\theta)\delta^2\delta'^2 z_{02} + (1+\theta)\delta^2\delta'^2 z_{12}\}] \\
& - \tfrac{1}{5040}\theta(1-\theta)(1+\theta)(2-\theta)(2+\theta)(3-\theta)[(4-\theta)\{(1-\chi)\delta^6 z_{00} + \chi\delta^6 z_{01}\} \\
& \qquad + (3+\theta)\{(1-\chi)\delta^6 z_{10} + \chi\delta^6 z_{11}\}] \\
& - \tfrac{1}{720}\chi(1-\chi)(2-\chi)(3-\chi)(4-\chi)(5-\chi)[(1-\theta)\delta'^6 z_{03} + \theta\delta'^6 z_{13}] \\
& - \tfrac{1}{240}\theta(1-\theta)\chi(1-\chi)(1+\theta)(2-\theta)[(3-\theta)\delta^4\delta'^2 z_{01} + (2+\theta)\delta^4\delta'^2 z_{11}] \\
& + \tfrac{1}{144}\theta(1-\theta)\chi(1-\chi)(2-\chi)(3-\chi)[(2-\theta)\delta'^4\delta^2 z_{02} + (1+\theta)\delta'^4\delta^2 z_{12}]
\end{aligned}$$

$$\ldots\ldots\ldots\ldots\text{(xliii)}.$$

With the usual notation:

$$\int_0^1\int_0^1 Z_0 d\theta d\chi = \tfrac{1}{4}(z_{00}+z_{01}+z_{10}+z_{11}) \quad\text{...............(xliv).}$$

$$\begin{aligned}\int_0^1\int_0^1 Z_2 d\theta d\chi &= \int_0^1 d\theta\,[-\tfrac{1}{12}\theta(1-\theta)(2-\theta)(\delta^2 z_{00}+\delta^2 z_{01}) - \tfrac{1}{12}\theta(1-\theta^2)(\delta^2 z_{10}+\delta^2 z_{11})\\ &\qquad -\tfrac{1}{8}\{(1-\theta)\,\delta'^2 z_{01}+\theta\delta'^2 z_{11}\} + \tfrac{1}{24}\{(1-\theta)\,\delta'^2 z_{02}+\theta\delta'^2 z_{12}\}]\\ &= [-\tfrac{1}{48}(\delta^2 z_{00}+\delta^2 z_{01}+\delta^2 z_{10}+\delta^2 z_{11}) - \tfrac{1}{16}(\delta'^2 z_{01}+\delta'^2 z_{11})\\ &\qquad + \tfrac{1}{48}(\delta'^2 z_{02}+\delta'^2 z_{12})] \quad\text{..............................(xlv).}\end{aligned}$$

$$\begin{aligned}\int_0^1\int_0^1 Z_4 d\theta d\chi &= \int_0^1 d\theta\,[\tfrac{1}{240}\theta(1-\theta^2)(2-\theta)(3-\theta)(\delta^4 z_{00}+\delta^4 z_{01})\\ &\qquad + \tfrac{1}{240}\theta(1-\theta^2)(4-\theta^2)(\delta^4 z_{10}+\delta^4 z_{11})\\ &\qquad + \tfrac{3}{160}\{(1-\theta)\,\delta'^4 z_{03}+\theta\delta'^4 z_{13}\} - \tfrac{13}{288}\{(1-\theta)\,\delta'^4 z_{02}+\theta\delta'^4 z_{12}\}]\\ &= [\tfrac{11}{2880}(\delta^4 z_{00}+\delta^4 z_{01}+\delta^4 z_{10}+\delta^4 z_{11}) + \tfrac{3}{320}(\delta'^4 z_{03}+\delta'^4 z_{13})\\ &\qquad - \tfrac{13}{576}(\delta'^4 z_{02}+\delta'^4 z_{12})\\ &\qquad + \tfrac{1}{576}\{3(\delta^2\delta'^2 z_{01}+\delta^2\delta'^2 z_{11}) - (\delta^2\delta'^2 z_{02}+\delta^2\delta'^2 z_{12})\}] \quad\text{...(xlvi).}\end{aligned}$$

$$\begin{aligned}\int_0^1\int_0^1 Z_6 d\theta d\chi &= \int_0^1 d\theta\,[-\tfrac{1}{10080}\theta(1-\theta^2)(4-\theta^2)(3-\theta)(4-\theta)(\delta^6 z_{00}+\delta^6 z_{01})\\ &\qquad - \tfrac{1}{10080}\theta(1-\theta^2)(4-\theta^2)(9-\theta^2)(\delta^6 z_{10}+\delta^6 z_{11})\\ &\qquad - \tfrac{863}{60480}\{(1-\theta)\,\delta'^6 z_{03}+\theta\delta'^6 z_{13}\}\\ &\qquad - \tfrac{1}{1440}\{\theta(1-\theta^2)(2-\theta)(3-\theta)\,\delta^4\delta'^2 z_{01}+\theta(1-\theta^2)\\ &\qquad\qquad \times(4-\theta^2)\,\delta^4\delta'^2 z_{11}\}\\ &\qquad + \tfrac{19}{4320}\{\theta(1-\theta)(2-\theta)\,\delta'^4\delta^2 z_{02}+\theta(1-\theta^2)\,\delta'^4\delta^2 z_{12}\}]\\ &= -\tfrac{1}{241920}[191(\delta^6 z_{00}+\delta^6 z_{01}+\delta^6 z_{10}+\delta^6 z_{11})\\ &\qquad + 1726(\delta'^6 z_{03}+\delta'^6 z_{13}) + 154(\delta^4\delta'^2 z_{01}+\delta^4\delta'^2 z_{11})\\ &\qquad - 266(\delta'^4\delta^2 z_{02}+\delta'^4\delta^2 z_{12})] \quad\text{.........................(xlvii).}\end{aligned}$$

Combining (xliv), (xlv), (xlvi) and (xlvii) we have

$$\begin{aligned}\int_0^1\int_0^1 z_{\theta,\chi}\,d\theta d\chi &= [\tfrac{1}{4}(z_{00}+z_{01}+z_{10}+z_{11}) - \tfrac{1}{48}(\delta^2 z_{00}+\delta^2 z_{01}+\delta^2 z_{10}+\delta^2 z_{11})\\ &\quad - \tfrac{1}{16}(\delta'^2 z_{01}+\delta'^2 z_{11}) + \tfrac{1}{48}(\delta'^2 z_{02}+\delta'^2 z_{12})\\ &\quad + \tfrac{11}{2880}(\delta^4 z_{00}+\delta^4 z_{01}+\delta^4 z_{10}+\delta^4 z_{11}) + \tfrac{3}{320}(\delta'^4 z_{03}+\delta'^4 z_{13})\\ &\quad - \tfrac{13}{576}(\delta'^4 z_{02}+\delta'^4 z_{12})\\ &\quad + \tfrac{1}{576}\{3(\delta^2\delta'^2 z_{01}+\delta^2\delta'^2 z_{11}) - (\delta^2\delta'^2 z_{02}+\delta^2\delta'^2 z_{12})\}\\ &\quad - \tfrac{1}{241920}\{191(\delta^6 z_{00}+\delta^6 z_{01}+\delta^6 z_{10}+\delta^6 z_{11}) + 1726(\delta'^6 z_{03}+\delta'^6 z_{13})\\ &\qquad + 154(\delta^4\delta'^2 z_{01}+\delta^4\delta'^2 z_{11}) - 266(\delta'^4\delta^2 z_{02}+\delta'^4\delta^2 z_{12})\}]\\ &\qquad\qquad\text{............(xlviii).}\end{aligned}$$

Expressing the differences in terms of the z's, we have the following expressions for $\int_0^h\int_0^k z\,dx\,dy$, correct, respectively, up to and including 1st, 3rd and 5th order differences:

$$\left\{\begin{array}{ll}
(\alpha) & \dfrac{hk}{4}(z_{00}+z_{01}+z_{10}+z_{11}),\\[2ex]
(\beta) & \dfrac{hk}{48}\left[\begin{array}{l} -z_{-10}+10z_{00}+10z_{10}-z_{20}\\ -z_{-11}+20z_{01}+20z_{11}-z_{21}\\ \quad -\ 5z_{02}-\ 5z_{12}\\ \quad +\ \ z_{03}+\ \ z_{13}\end{array}\right],\\[2ex]
(\gamma) & \dfrac{hk}{2880}\left[\begin{array}{l} 11z_{-20}-78z_{-10}+542z_{00}+542z_{10}-78z_{20}+11z_{30}\\ +11z_{-21}-128z_{-11}+1544z_{01}+1544z_{11}-128z_{21}+11z_{31}\\ \quad +25z_{-12}-823z_{02}-823z_{12}+25z_{22}\\ \quad -5z_{-13}+487z_{03}+487z_{13}-5z_{23}\\ \quad\quad -173z_{04}-173z_{14}\\ \quad\quad +27z_{05}+27z_{15}\end{array}\right]
\end{array}\right. \quad \ldots\text{(xlix)}.$$

Second Panel-Ordinary.

The formula is:

$$\begin{aligned}
z_{\theta,\chi} &= (1-\chi)(1-\theta)z_{01}+(1-\chi)\theta z_{11}+\chi(1-\theta)z_{02}+\chi\theta z_{12}\\
&\quad -\tfrac{1}{6}\chi(1-\chi)[(2-\chi)\{(1-\theta)\delta'^2 z_{01}+\theta\delta'^2 z_{11}\}+(1+\chi)\{(1-\theta)\delta'^2 z_{02}+\theta\delta'^2 z_{12}\}]\\
&\quad -\tfrac{1}{6}\theta(1-\theta)[(2-\theta)\{(1-\chi)\delta^2 z_{01}+\chi\delta^2 z_{02}\}+(1+\theta)\{(1-\chi)\delta^2 z_{11}+\chi\delta^2 z_{12}\}]\\
&\quad +\tfrac{1}{120}\chi(1-\chi)(1+\chi)(2-\chi)[(8-\chi)\{(1-\theta)\delta'^4 z_{02}+\theta\delta'^4 z_{12}\}\\
&\qquad\qquad -(3-\chi)\{(1-\theta)\delta'^4 z_{03}+\theta\delta'^4 z_{13}\}]\\
&\quad +\tfrac{1}{120}\theta(1-\theta)(1+\theta)(2-\theta)[(3-\theta)\{(1-\chi)\delta^4 z_{01}+\chi\delta^4 z_{02}\}\\
&\qquad\qquad +(2+\theta)\{(1-\chi)\delta^4 z_{11}+\chi\delta^4 z_{12}\}]\\
&\quad +\tfrac{1}{36}\theta(1-\theta)\chi(1-\chi)[(2-\theta)\{(2-\chi)\delta^2\delta'^2 z_{01}+(1+\chi)\delta^2\delta'^2 z_{02}\}\\
&\qquad\qquad +(1+\theta)\{(2-\chi)\delta^2\delta'^2 z_{11}+(1+\chi)\delta^2\delta'^2 z_{12}\}]\\
&\quad -\tfrac{1}{5040}\theta(1-\theta)(1+\theta)(2-\theta)(2+\theta)(3-\theta)[(4-\theta)\{(1-\chi)\delta^6 z_{01}+\chi\delta^6 z_{02}\}\\
&\qquad\qquad +(3+\theta)\{(1-\chi)\delta^6 z_{11}+\chi\delta^6 z_{12}\}]\\
&\quad +\tfrac{1}{720}\chi(1-\chi)(1+\chi)(2-\chi)(3-\chi)(4-\chi)[(1-\theta)\delta'^6 z_{03}+\theta\delta'^6 z_{13}]\\
&\quad -\tfrac{1}{144}\theta(1-\theta)\chi(1-\chi)(1+\chi)(2-\chi)[(2-\theta)\delta^2\delta'^4 z_{02}+(1+\theta)\delta^2\delta'^4 z_{12}]\\
&\quad -\tfrac{1}{720}\theta(1-\theta)\chi(1-\chi)(1+\theta)(2-\theta)[(3-\theta)\{(2-\chi)\delta^4\delta'^2 z_{01}+(1+\chi)\delta^4\delta'^2 z_{02}\}\\
&\qquad\qquad +(2+\theta)\{(2-\chi)\delta^4\delta'^2 z_{11}+(1+\chi)\delta^4\delta'^2 z_{12}\}]\ldots\text{(l)}.
\end{aligned}$$

We have $$\int_0^1\int_0^1 Z_0 d\theta d\chi = \tfrac{1}{4}(z_{01}+z_{11}+z_{12}+z_{02}) \quad \text{(li)}.$$

$$\begin{aligned}\int_0^1\int_0^1 Z_2 d\theta d\chi = \int_0^1 d\theta\,[&-\tfrac{1}{24}\{(1-\theta)\,\delta'^2 z_{01}+\theta\delta'^2 z_{11}\}-\tfrac{1}{24}\{(1-\theta)\,\delta'^2 z_{02}+\theta\delta'^2 z_{12}\}\\ &-\tfrac{1}{12}\theta(1-\theta)(2-\theta)(\delta^2 z_{01}+\delta^2 z_{02})-\tfrac{1}{12}\theta(1-\theta^2)(\delta^2 z_{11}+\delta^2 z_{12})]\\ = -\tfrac{1}{48}(&\delta'^2 z_{01}+\delta'^2 z_{11}+\delta'^2 z_{02}+\delta'^2 z_{12}+\delta^2 z_{01}+\delta^2 z_{02}+\delta^2 z_{11}+\delta^2 z_{12})\end{aligned} \quad \text{(lii)}.$$

$$\begin{aligned}\int_0^1\int_0^1 Z_4 d\theta d\chi = \int_0^1 d\theta\,[&\tfrac{11}{480}\{(1-\theta)\,\delta'^4 z_{02}+\theta\delta'^4 z_{12}\}-\tfrac{11}{1440}\{(1-\theta)\,\delta'^4 z_{03}+\theta\delta'^4 z_{13}\}\\ &+\tfrac{1}{120}\theta(1-\theta^2)(2-\theta)(3-\theta)\,\tfrac{1}{2}(\delta^4 z_{01}+\delta^4 z_{02})\\ &+\tfrac{1}{120}\theta(1-\theta^2)(4-\theta^2)\,\tfrac{1}{2}(\delta^4 z_{11}+\delta^4 z_{12})\\ &+\tfrac{1}{144}\theta(1-\theta)(2-\theta)\{\delta^2\delta'^2 z_{01}+\delta^2\delta'^2 z_{02}\}\\ &+\tfrac{1}{144}\theta(1-\theta^2)\{\delta^2\delta'^2 z_{11}+\delta^2\delta'^2 z_{12}\}\\ =[\tfrac{11}{2880}\{3(&\delta'^4 z_{02}+\delta'^4 z_{12})-(\delta'^4 z_{03}+\delta'^4 z_{13})+\delta^4 z_{01}+\delta^4 z_{02}+\delta^4 z_{11}+\delta^4 z_{12}\}\\ &+\tfrac{1}{576}\{\delta^2\delta'^2 z_{01}+\delta^2\delta'^2 z_{02}+\delta^2\delta'^2 z_{11}+\delta^2\delta'^2 z_{12}\}]\end{aligned} \quad \text{(liii)}.$$

$$\begin{aligned}\int_0^1\int_0^1 Z_6 d\theta d\chi = \int_0^1 d\theta\,[&-\tfrac{1}{10080}\theta(1-\theta^2)(4-\theta^2)(3-\theta)(4-\theta)(\delta^6 z_{01}+\delta^6 z_{02})\\ &-\tfrac{1}{10080}\theta(1-\theta^2)(4-\theta^2)(9-\theta^2)(\delta^6 z_{11}+\delta^6 z_{12})\\ &+\tfrac{271}{60480}\{(1-\theta)\,\delta'^6 z_{03}+\theta\delta'^6 z_{13}\}\\ &-\tfrac{11}{4320}\{\theta(1-\theta)(2-\theta)\,\delta^2\delta'^4 z_{02}+\theta(1-\theta^2)\,\delta^2\delta'^4 z_{12}\}\\ &-\tfrac{1}{2880}\{\theta(1-\theta^2)(2-\theta)(3-\theta)(\delta^4\delta'^2 z_{01}+\delta^4\delta'^2 z_{02})\\ &\qquad+\theta(1-\theta^2)(4-\theta^2)(\delta^4\delta'^2 z_{11}+\delta^4\delta'^2 z_{12})\}]\\ =\tfrac{1}{241920}[&-191(\delta^6 z_{01}+\delta^6 z_{02}+\delta^6 z_{11}+\delta^6 z_{12})+542(\delta'^6 z_{03}+\delta'^6 z_{13})\\ &-154(\delta^2\delta'^4 z_{02}+\delta^2\delta'^4 z_{12})\\ &-77(\delta^4\delta'^2 z_{01}+\delta^4\delta'^2 z_{02}+\delta^4\delta'^2 z_{11}+\delta^4\delta'^2 z_{12})]\end{aligned} \quad \text{(liv)}.$$

Combining (li), (lii), (liii) and (liv) we have

$$\begin{aligned}\int_0^1\int_0^1 z_{\theta,\chi} d\theta d\chi = [&\tfrac{1}{4}(z_{01}+z_{02}+z_{11}+z_{12})\\ &-\tfrac{1}{48}(\delta^2 z_{01}+\delta^2 z_{02}+\delta^2 z_{11}+\delta^2 z_{12}+\delta'^2 z_{01}+\delta'^2 z_{02}+\delta'^2 z_{11}+\delta'^2 z_{12})\\ &+\tfrac{11}{2880}\{3(\delta'^4 z_{02}+\delta'^4 z_{12})-(\delta'^4 z_{03}+\delta'^4 z_{13})+\delta^4 z_{01}+\delta^4 z_{02}+\delta^4 z_{11}+\delta^4 z_{12}\}\\ &+\tfrac{1}{576}\{\delta^2\delta'^2 z_{01}+\delta^2\delta'^2 z_{02}+\delta^2\delta'^2 z_{11}+\delta^2\delta'^2 z_{12}\}\\ &+\tfrac{1}{241920}\{-191(\delta^6 z_{01}+\delta^6 z_{11}+\delta^6 z_{02}+\delta^6 z_{12})+542(\delta'^6 z_{03}+\delta'^6 z_{13})\\ &\qquad-154(\delta^2\delta'^4 z_{02}+\delta^2\delta'^4 z_{12})\\ &\qquad-77(\delta^4\delta'^2 z_{01}+\delta^4\delta'^2 z_{02}+\delta^4\delta'^2 z_{11}+\delta^4\delta'^2 z_{12})\}]\end{aligned} \quad \text{(lv)}.$$

Expressing the differences in terms of the z's, we have the following three expressions for $\int_0^h \int_k^{2k} z\,dx\,dy$, respectively, correct up to and including 1st, 3rd and 5th order differences:

$$\left\{\begin{array}{ll}
(\alpha) & \dfrac{hk}{4}(z_{01}+z_{11}+z_{12}+z_{02}), \\[2ex]
(\beta) & \dfrac{hk}{48}\left[\begin{array}{c} -\ z_{00} -\ z_{10} \\ -z_{-11}+14z_{01}+14z_{11}-z_{21} \\ -z_{-12}+14z_{02}+14z_{12}-z_{22} \\ -\ z_{03}-\ z_{13} \end{array}\right], \\[2ex]
(\gamma) & \dfrac{hk}{2880}\left[\begin{array}{c} 5z_{-10}-\ 32z_{00}-\ 32z_{10}+\ 5z_{20} \\ 11z_{-21}-98z_{-11}+\ 724z_{01}+\ 724z_{11}-98z_{21}+11z_{31} \\ +11z_{-22}-98z_{-12}+1109z_{02}+1109z_{12}-98z_{22}+11z_{32} \\ +\ 5z_{-13}-\ 263z_{03}-\ 263z_{13}+\ 5z_{23} \\ +\ 77z_{04}+\ 77z_{14} \\ -\ 11z_{05}-\ 11z_{15} \end{array}\right]
\end{array}\right. \quad \text{...(lvi).}$$

'Ordinary First Panel,' 'Ordinary-Second Panel' and 'Second Panel-First Panel' formulae can clearly be obtained by interchanges of suffixes in the formulae already established.

We now give the additional boundary formulae that are required when it is desired to work to 6th differences. They were obtained by a direct extension of the univariate mid-panel formulae given in Part I.

Third Panel-Ordinary.

$$\begin{aligned}
\int_0^h \int_{2k}^{3k} z\,dx\,dy = hk\,[\tfrac{1}{4}(z_{02}+z_{03}+z_{12}+z_{13}) &- \tfrac{1}{48}(\delta^2 z_{02}+\delta^2 z_{03}+\delta^2 z_{12}+\delta^2 z_{13}) \\
&- \tfrac{1}{48}(\delta'^2 z_{02}+\delta'^2 z_{03}+\delta'^2 z_{12}+\delta'^2 z_{13}) \\
&+ \tfrac{1}{576}(\delta^2\delta'^2 z_{02}+\delta^2\delta'^2 z_{03}+\delta^2\delta'^2 z_{12}+\delta^2\delta'^2 z_{13}) \\
&+ \tfrac{11}{2880}(\delta^4 z_{02}+\delta^4 z_{03}+\delta^4 z_{12}+\delta^4 z_{13}+\delta'^4 z_{02}+\delta'^4 z_{03}+\delta'^4 z_{12}+\delta'^4 z_{13}) \\
&- \tfrac{11}{34560}(\delta^2\delta'^4 z_{02}+\delta^2\delta'^4 z_{03}+\delta^2\delta'^4 z_{12}+\delta^2\delta'^4 z_{13}+\delta^4\delta'^2 z_{02} \\
&\qquad +\delta^4\delta'^2 z_{03}+\delta^4\delta'^2 z_{12}+\delta^4\delta'^2 z_{13}) \\
&- \tfrac{191}{241920}\{2(\delta'^6 z_{03}+\delta'^6 z_{13})+\delta^6 z_{02}+\delta^6 z_{03}+\delta^6 z_{12}+\delta^6 z_{13}\}]
\end{aligned}$$

............(lvii)

(correct up to and including 6th order differences).

Third Panel-Third Panel.

$$\int_{2h}^{3h}\int_{2k}^{3k} z\,dx\,dy = hk\,[\tfrac{1}{4}(z_{22}+z_{23}+z_{32}+z_{33}) - \tfrac{1}{48}(\delta^2 z_{22}+\delta^2 z_{23}+\delta^2 z_{32}+\delta^2 z_{33})$$
$$-\tfrac{1}{48}(\delta'^2 z_{22}+\delta'^2 z_{23}+\delta'^2 z_{32}+\delta'^2 z_{33})$$
$$+\tfrac{1}{576}(\delta^2\delta'^2 z_{22}+\delta^2\delta'^2 z_{23}+\delta^2\delta'^2 z_{32}+\delta^2\delta'^2 z_{33})$$
$$+\tfrac{11}{2880}\{\delta^4 z_{22}+\delta^4 z_{23}+\delta^4 z_{32}+\delta^4 z_{33}+\delta'^4 z_{22}+\delta'^4 z_{23}+\delta'^4 z_{32}+\delta'^4 z_{33}\}$$
$$-\tfrac{11}{34560}\{\delta^2\delta'^4 z_{22}+\delta^2\delta'^4 z_{23}+\delta^2\delta'^4 z_{32}+\delta^2\delta'^4 z_{33}+\delta^4\delta'^2 z_{22}$$
$$+\delta^4\delta'^2 z_{23}+\delta^4\delta'^2 z_{32}+\delta^4\delta'^2 z_{33}\}$$
$$-\tfrac{191}{241920}\{2(\delta'^6 z_{23}+\delta'^6 z_{33}+\delta^6 z_{32}+\delta^6 z_{33})\}] \quad \ldots\ldots\ldots\ldots(\text{lviii})$$

(correct up to and including 6th order differences).

Third Panel-Second Panel.

$$\int_{h}^{2h}\int_{2k}^{3k} z\,dx\,dy = hk\,[\tfrac{1}{4}(z_{12}+z_{13}+z_{22}+z_{23}) - \tfrac{1}{48}(\delta^2 z_{12}+\delta^2 z_{13}+\delta^2 z_{22}+\delta^2 z_{23})$$
$$-\tfrac{1}{48}(\delta'^2 z_{12}+\delta'^2 z_{13}+\delta'^2 z_{22}+\delta'^2 z_{23})$$
$$+\tfrac{1}{576}(\delta^2\delta'^2 z_{12}+\delta^2\delta'^2 z_{13}+\delta^2\delta'^2 z_{22}+\delta^2\delta'^2 z_{23})$$
$$+\tfrac{11}{2880}\{\delta'^4 z_{12}+\delta'^4 z_{13}+\delta'^4 z_{22}+\delta'^4 z_{23}+3(\delta^4 z_{22}+\delta^4 z_{23})-\delta^4 z_{32}-\delta^4 z_{33}\}$$
$$-\tfrac{11}{34560}\{\delta^2\delta'^4 z_{12}+\delta^2\delta'^4 z_{13}+\delta^2\delta'^4 z_{22}+\delta^2\delta'^4 z_{23}+2(\delta^4\delta'^2 z_{22}+\delta^4\delta'^2 z_{23})\}$$
$$+\tfrac{1}{241920}\{542(\delta^6 z_{32}+\delta^6 z_{33})-382(\delta'^6 z_{13}+\delta'^6 z_{23})\}] \quad \ldots(\text{lix})$$

(correct up to and including 6th order differences).

Third Panel-First Panel.

$$\int_{0}^{h}\int_{2k}^{3k} z\,dx\,dy = hk\,[\tfrac{1}{4}(z_{02}+z_{03}+z_{12}+z_{13})$$
$$-\tfrac{1}{48}\{\delta'^2 z_{02}+\delta'^2 z_{03}+\delta'^2 z_{12}+\delta'^2 z_{13}+3(\delta^2 z_{12}+\delta^2 z_{13})-\delta^2 z_{22}-\delta^2 z_{23}\}$$
$$+\tfrac{1}{2880}\{11(\delta'^4 z_{02}+\delta'^4 z_{03}+\delta'^4 z_{12}+\delta'^4 z_{13})-65(\delta^4 z_{22}+\delta^4 z_{23})$$
$$+27(\delta^4 z_{32}+\delta^4 z_{33})\}$$
$$+\tfrac{1}{576}\{3(\delta^2\delta'^2 z_{12}+\delta^2\delta'^2 z_{13})-\delta^2\delta'^2 z_{22}-\delta^2\delta'^2 z_{23}\}$$
$$+\tfrac{1}{34560}\{38(\delta^4\delta'^2 z_{22}+\delta^4\delta'^2 z_{23})-22(\delta^2\delta'^4 z_{12}+\delta^2\delta'^4 z_{13})\}$$
$$-\tfrac{1}{241920}\{382(\delta'^6 z_{03}+\delta'^6 z_{13})+1726(\delta^6 z_{32}+\delta^6 z_{33})\}] \quad \ldots(\text{lx})$$

(correct up to and including 6th order differences).

These formulae are given in terms of ordinates only in Table IV. They are all we require, since the remainder can be obtained by interchanging suffixes in those already established.

We are now in a position to proceed to the final stage of the work and to find approximations for $\int_0^{mh}\int_0^{nk} z\,dx\,dy$, depending only on ordinates within the rectangle $x=0$ to mh, $y=0$ to nk. We have only to associate with every panel its appropriate cubature formula already obtained and to sum the results.

Furthermore the results obtained will, by a change of origin, be applicable to any double integral in which the limits are constants, or, in other words, whenever we are integrating over a rectangle.

General formulae can only be obtained if the number of ordinates in each side of the rectangle exceeds 8 in the case of the third difference formula and 12 in the case of the fifth difference formula; and these formulae are given in the appended Table V; when we have less than this number of ordinates, the formulae for each panel 'overlap' differently in each case considered; for example, in the case of 16 ordinates we have the following formula correct up to and including third order differences:

$$\int_0^{3h}\int_0^{3k} z\,dx\,dy = \frac{hk}{16}\left\{\begin{array}{l} 2z_{00} + 7z_{10} + 7z_{20} + 2z_{30} \\ + 7z_{01} + 20z_{11} + 20z_{21} + 7z_{31} \\ + 7z_{02} + 20z_{12} + 20z_{22} + 7z_{32} \\ + 2z_{03} + 7z_{13} + 7z_{23} + 2z_{33} \end{array}\right\} \quad \text{.........(lxi)},$$

but if we take 25 ordinates we have, correct up to and including 3rd order differences,

$$\int_0^{4h}\int_0^{4k} z\,dx\,dy = \frac{hk}{12}\left\{\begin{array}{l} z_{00} + 6z_{10} + 2z_{20} + 6z_{30} + z_{40} \\ + 6z_{01} + 20z_{11} + 12z_{21} + 20z_{31} + 6z_{41} \\ + 2z_{02} + 12z_{12} + 4z_{22} + 12z_{32} + 2z_{42} \\ + 6z_{03} + 20z_{13} + 12z_{23} + 20z_{33} + 6z_{43} \\ + z_{04} + 6z_{14} + 2z_{24} + 6z_{34} + z_{44} \end{array}\right\} \quad \text{...(lxii)}.$$

The computer will easily be able to derive similar formulae for himself from the formulae we have given for the separate panels if he does not desire to use as many ordinates as the general formulae necessitate; though in some cases the accuracy, that his work demands, will compel him to do so.

We will now consider a numerical example, namely

$$W = \frac{1}{2\pi}\frac{1}{\sqrt{1-r^2}}\int_0^h\int_0^k e^{-\frac{1}{2}\frac{1}{1-r^2}(x^2-2rxy+y^2)}\,dx\,dy,$$

which we will compute for $h=1$, $k=\cdot 8$, $r=\cdot 2$. We have chosen this integral because the result may be checked by a computation from its expansion in tetrachoric functions.

Formula B of Table V will be used. We give below a table of ordinates of

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{1}{(1-r^2)}(x^2-2rxy+y^2)}$$

for $r = \cdot 2$; each with its appropriate multiplier taken from the formula:

TABLE VI. *Table of* $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{1\cdot 92}(x^2-\cdot 4xy+y^2)}$.

x / y	0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1·0
·8	·28585 4	·28915 23	·28945 12	·28675 17	·28113 16	·27276 16	·26191 16	·24887 17	·23404 12	·21781 23	·20060 4
·7	·30908 23	·31199 76	·31167 54	·30812 64	·30145 62	·29187 62	·27967 62	·26520 64	·24887 54	·23113 76	·21243 23
·6	·33073 12	·33315 54	·33211 32	·32765 42	·31989 40	·30908 40	·29554 40	·27967 42	·26191 32	·24273 54	·22262 12
·5	·35024 17	·35207 64	·35024 42	·34481 52	·33594 50	·32391 50	·30908 50	·29187 52	·27276 42	·25227 64	·23089 17
·4	·36704 16	·36819 62	·36552 40	·35910 50	·34914 48	·33594 48	·31989 48	·30145 50	·28113 40	·25946 62	·23698 16
·3	·38067 17	·38107 64	·37751 42	·37011 52	·35910 50	·34481 50	·32765 50	·30812 52	·28675 42	·26410 64	·24071 17
·2	·39071 12	·39031 54	·38586 32	·37751 42	·36552 40	·35024 40	·33211 40	·31167 42	·28945 32	·26603 54	·24197 12
·1	·39687 23	·39563 76	·39031 54	·38107 64	·36819 62	·35207 62	·33315 62	·31199 64	·28915 54	·26520 76	·24071 23
0	·39894 4	·39687 23	·39071 12	·38067 17	·36704 16	·35024 16	·33073 16	·30908 17	·28585 12	·26163 23	·23698 4

We find $$\frac{1}{\sqrt{2\pi}}\int_0^1\int_0^{\cdot 8} e^{-\frac{1}{1\cdot 92}(x^2-\cdot 4xy+y^2)}\,dx\,dy = \cdot 25318(29),$$

whence $$W = \frac{1}{2\pi}\frac{1}{\sqrt{\cdot 96}}\int_0^1\int_0^{\cdot 8} e^{-\frac{1}{1\cdot 92}(x^2-\cdot 4xy+y^2)}\,dx\,dy = \cdot 10309 \quad \ldots\text{(lxiii)}.$$

The expansion of W in tetrachoric functions is

$$W = \{\tfrac{1}{2}-\tau_0(h)\}\{\tfrac{1}{2}-\tau_0(k)\} + \{\tau_1(0)-\tau_1(h)\}\{\tau_1(0)-\tau_1(k)\}r + \ldots$$
$$+ \{\tau_n(0)-\tau_n(h)\}\{\tau_n(0)-\tau_n(k)\}r^n + \ldots,$$

where $$\tau_n(x) = \frac{1}{\sqrt{\lfloor n}}\left(-\frac{d}{dx}\right)^{n-1}\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(lxiv)}.$$

We have the following set of numerical values*:

n	$\tau_n(0)$	$\tau_n(1)$	$\tau_n(\cdot 8)$	$\tau_n(0)-\tau_n(\cdot 8)$	$\tau_n(0)-\tau_n(1)$
0	·5	·15867	·21186	·28814	·34133
1	·39894	·24198	·28970	·10924	·15696
2	0	·17110	·16388	− ·16388	− ·17110
3	− ·16287	0	− ·04258	− ·12029	− ·16287
4	0	− ·09878	− ·11165	− ·11165	·09878
5	·10925	− ·04418	− ·01138	·12063	·15343
6	0	05411	·07782	− ·07782	− ·05411

n	$\{\tau_n(0)-\tau_n(1)\}\{\tau_n(0)-\tau_n(\cdot 8)\}$	r^n	$\{\tau_n(0)-\tau_n(1)\}\{\tau_n(0)-\tau_n(\cdot 8)\}r^n$
0	·09835	1	·09835
1	·01715	·2	·00343
2	·02804	·04	·00112
3	·01959	·008	·00016
4	·01103	·0016	·00002
5	·01851	·00032	·00001
6	·00421	·000064	0
			$W=\cdot 10309$

We see that the results of the two methods are in agreement to five figures. We have only worked to five figures because tables of the tetrachoric functions were only available to this degree of accuracy.

We will now proceed to an example which illustrates the possibility, where accuracy to a great many figures is not necessary, of obtaining quite good results from a simple formula involving comparatively few ordinates.

In a recent paper† Mr E. C. Rhodes fitted a skew-correlation surface to some barometric data for Laudale and Southampton.

The form of his surface is

$$z = z_0 e^{-(lx+my)}\left(1-\frac{x}{a}+\frac{y}{b}\right)^p\left(1+\frac{x}{a'}-\frac{y}{b'}\right)^{p'},$$

the numerical value of the constants for the barometric data being given in his paper.

We have calculated the ordinates at the 16 points corresponding to barometric pressures of 29·95″, 30·05″, 30·15″ and 30·25″ at Laudale, and 30·05″, 30·15″, 30·25″ and 30·35″ at Southampton.

* Tables of the tetrachoric functions are given in *Tables for Statisticians*, Table XXIX.

† *Biometrika*, Vol. XIV, p. 375.

We find:

		Southampton			
		30·35″	30·25″	30·15″	30·05″
Laudale	30·25″	46·09	48·82	37·44	21·92
	30·15″	42·57	57·88	55·46	40·19
	30·05″	30·71	53·96	64·11	56·41
	29·95″	17·51	40·57	60·28	63·67

We use the formula of (lxi) to calculate the portion of the frequency within this rectangle and find for its value 464·1.

Mr Rhodes has calculated the ordinates at the middle points of the cells bounded by the above ordinates, and from these, by an entirely different method, found for the frequency 462·8. This is quite good agreement considering the nature of the approximations employed.

D. Mid-Ordinate Cubature Formulae for $\int_0^{mh}\int_0^{nk} z\,dx\,dy$.

In this section we will give, briefly, the alternative formulae that are necessary, if it is desired to base the cubature on the centre ordinate of each panel instead of on the corner ordinates as we have done hitherto. The necessary formulae are easily derived from the mid-ordinate, univariate formulae of Part I and, if it is necessary to work to 6th differences and, at the same time, to avoid using ordinates outside the rectangle of integration we shall require as before 10 formulae. As the method of obtaining each one is the same, we will, for the sake of brevity, give the proof only for the first formula and state the remainder.

The Ordinary Formula (*not near a boundary*). Let $z_{\frac{1}{2},\frac{1}{2}}$ be the centre ordinate of the panel bounded by z_{00}, z_{01}, z_{10}, z_{11} and let $Z_{\frac{1}{2}} = \int_0^k z_{\frac{1}{2},y}\,dy$. Then by formula (xi) of Part I we have

$$\int_0^h\int_0^k z\,dx\,dy = h\left[Z_{\frac{1}{2}} + \tfrac{1}{24}\delta^2 Z_{\frac{1}{2}} - \tfrac{17}{5760}\delta^4 Z_{\frac{1}{2}} + \tfrac{367}{967680}\delta^6 Z_{\frac{1}{2}} + \dots\right],$$

$$Z_{\frac{1}{2}} = k\left[z_{\frac{1}{2},\frac{1}{2}} + \tfrac{1}{24}\delta'^2 z_{\frac{1}{2},\frac{1}{2}} - \tfrac{17}{5760}\delta'^4 z_{\frac{1}{2},\frac{1}{2}} + \tfrac{367}{967680}\delta'^6 z_{\frac{1}{2},\frac{1}{2}} + \dots\right],$$

whence, neglecting 7th* and higher order differences,

$$\int_0^h\int_0^k z\,dx\,dy = hk\left[z_{\frac{1}{2},\frac{1}{2}} + \tfrac{1}{24}(\delta^2 z_{\frac{1}{2},\frac{1}{2}} + \delta'^2 z_{\frac{1}{2},\frac{1}{2}}) - \tfrac{17}{5760}(\delta^4 z_{\frac{1}{2},\frac{1}{2}} + \delta'^4 z_{\frac{1}{2},\frac{1}{2}}) + \tfrac{1}{576}\delta^2\delta'^2 z_{\frac{1}{2},\frac{1}{2}} \right.$$
$$\left. - \tfrac{17}{138240}(\delta^4\delta'^2 z_{\frac{1}{2},\frac{1}{2}} + \delta^2\delta'^4 z_{\frac{1}{2},\frac{1}{2}}) + \tfrac{367}{967680}(\delta'^6 z_{\frac{1}{2},\frac{1}{2}} + \delta^6 z_{\frac{1}{2},\frac{1}{2}})\right].$$

* In this particular formula the 7th order differences vanish, but in the following formulae they do not, but are neglected.

Boundary Formulae. The formulae necessary are as follows:

First Panel-Ordinary.

$$\begin{aligned}\int_0^h\int_0^k z\,dx\,dy = hk\,[&z_{\frac12,\frac12}+\tfrac{1}{24}(\delta^2 z_{\frac12,\frac12}+2\delta'^2 z_{\frac12,\frac32}-\delta'^2 z_{\frac12,\frac52})\\&+\tfrac{1}{5760}(429\delta'^4 z_{\frac12,\frac52}-206\delta'^4 z_{\frac12,\frac72}-17\delta^4 z_{\frac12,\frac12})\\&+\tfrac{1}{576}(2\delta^2\delta'^2 z_{\frac12,\frac32}-\delta^2\delta'^2 z_{\frac12,\frac52})+\tfrac{1}{138240}(223\delta^2\delta'^4 z_{\frac12,\frac52}-17\delta^4\delta'^2 z_{\frac12,\frac32})\\&+\tfrac{1}{967680}(32119\delta'^6 z_{\frac12,\frac72}+367\delta^6 z_{\frac12,\frac12})].\end{aligned}$$

Second Panel-Ordinary.

$$\begin{aligned}\int_0^h\int_k^{2k} z\,dx\,dy = hk\,[&z_{\frac12,\frac32}+\tfrac{1}{24}(\delta^2 z_{\frac12,\frac32}+\delta'^2 z_{\frac12,\frac32})-\tfrac{17}{5760}(2\delta'^4 z_{\frac12,\frac52}-\delta'^4 z_{\frac12,\frac72}+\delta^4 z_{\frac12,\frac32})\\&+\tfrac{1}{576}\delta^2\delta'^2 z_{\frac12,\frac32}-\tfrac{17}{138240}(\delta^2\delta'^4 z_{\frac12,\frac52}+\delta^4\delta'^2 z_{\frac12,\frac32})\\&+\tfrac{1}{967680}(367\delta^6 z_{\frac12,\frac32}-2489\delta'^6 z_{\frac12,\frac72})].\end{aligned}$$

Third Panel-Ordinary.

$$\begin{aligned}\int_0^h\int_{2k}^{3k} z\,dx\,dy = hk[&z_{\frac12,\frac52}+\tfrac{1}{24}(\delta^2 z_{\frac12,\frac52}+\delta'^2 z_{\frac12,\frac52})-\tfrac{17}{5760}(\delta'^4 z_{\frac12,\frac52}+\delta^4 z_{\frac12,\frac52})+\tfrac{1}{576}\delta^2\delta'^2 z_{\frac12,\frac52}\\&-\tfrac{17}{138240}(\delta^2\delta'^4 z_{\frac12,\frac52}+\delta^4\delta'^2 z_{\frac12,\frac52})+\tfrac{367}{967680}(\delta^6 z_{\frac12,\frac52}+\delta'^6 z_{\frac12,\frac72})].\end{aligned}$$

First Panel-First Panel.

$$\begin{aligned}\int_0^h\int_0^k z\,dx\,dy = hk\,[&z_{\frac12,\frac12}+\tfrac{1}{24}(2\delta'^2 z_{\frac12,\frac32}-\delta'^2 z_{\frac12,\frac52}+2\delta^2 z_{\frac32,\frac12}-\delta^2 z_{\frac52,\frac12})\\&+\tfrac{1}{5760}\{429(\delta'^4 z_{\frac12,\frac52}+\delta^4 z_{\frac52,\frac12})-206(\delta'^4 z_{\frac12,\frac72}+\delta^4 z_{\frac72,\frac12})\}\\&+\tfrac{1}{576}(4\delta^2\delta'^2 z_{\frac32,\frac32}-2\delta^2\delta'^2 z_{\frac32,\frac52}-2\delta^2\delta'^2 z_{\frac52,\frac32}+\delta^2\delta'^2 z_{\frac52,\frac52})\\&+\tfrac{223}{138240}(\delta^2\delta'^4 z_{\frac32,\frac52}+\delta^4\delta'^2 z_{\frac52,\frac32})+\tfrac{32119}{967680}(\delta'^6 z_{\frac12,\frac72}+\delta^6 z_{\frac72,\frac12})].\end{aligned}$$

First Panel-Second Panel.

$$\begin{aligned}\int_h^{2h}\int_0^k z\,dx\,dy = hk\,[&z_{\frac32,\frac12}+\tfrac{1}{24}(2\delta'^2 z_{\frac32,\frac32}-\delta'^2 z_{\frac32,\frac52}+\delta^2 z_{\frac32,\frac12})\\&+\tfrac{1}{5760}(429\delta'^4 z_{\frac32,\frac52}-206\delta'^4 z_{\frac32,\frac72}-34\delta^4 z_{\frac52,\frac12}+17\delta^4 z_{\frac72,\frac12})\\&+\tfrac{1}{576}(2\delta^2\delta'^2 z_{\frac32,\frac32}-\delta^2\delta'^2 z_{\frac32,\frac52})+\tfrac{1}{138240}(223\delta^2\delta'^4 z_{\frac32,\frac52}-17\delta^4\delta'^2 z_{\frac52,\frac32})\\&+\tfrac{1}{967680}(32119\delta'^6 z_{\frac32,\frac72}-2489\delta^6 z_{\frac72,\frac12})].\end{aligned}$$

Second Panel-Second Panel.

$$\begin{aligned}\int_h^{2h}\int_k^{2k} z\,dx\,dy = hk\,[&z_{\frac32,\frac32}+\tfrac{1}{24}(\delta^2 z_{\frac32,\frac32}+\delta'^2 z_{\frac32,\frac32})\\&-\tfrac{17}{5760}(2\delta'^4 z_{\frac32,\frac52}-\delta'^4 z_{\frac32,\frac72}+2\delta^4 z_{\frac52,\frac32}-\delta^4 z_{\frac72,\frac32})+\tfrac{1}{576}\delta^2\delta'^2 z_{\frac32,\frac32}\\&-\tfrac{17}{138240}(\delta^2\delta'^4 z_{\frac32,\frac52}+\delta^4\delta'^2 z_{\frac52,\frac32})-\tfrac{2489}{967680}(\delta'^6 z_{\frac32,\frac72}+\delta^6 z_{\frac72,\frac32})].\end{aligned}$$

Third Panel-First Panel.

$$\int_0^h \int_{2k}^{3k} z\,dx\,dy = hk\,[z_{\frac{1}{2},\frac{5}{2}} + \tfrac{1}{24}(\delta'^2 z_{\frac{1}{2},\frac{5}{2}} + 2\delta^2 z_{\frac{3}{2},\frac{5}{2}} - \delta^2 z_{\frac{5}{2},\frac{5}{2}})$$
$$+ \tfrac{1}{5760}(429\delta^4 z_{\frac{5}{2},\frac{5}{2}} - 206\delta^4 z_{\frac{7}{2},\frac{5}{2}} - 17\delta'^4 z_{\frac{1}{2},\frac{5}{2}})$$
$$+ \tfrac{1}{576}(2\delta^2\delta'^2 z_{\frac{3}{2},\frac{5}{2}} - \delta^2\delta'^2 z_{\frac{5}{2},\frac{5}{2}})$$
$$+ \tfrac{1}{138240}(223\delta^4\delta'^2 z_{\frac{5}{2},\frac{5}{2}} - 17\delta^2\delta'^4 z_{\frac{3}{2},\frac{5}{2}})$$
$$+ \tfrac{1}{967680}(367\delta'^6 z_{\frac{1}{2},\frac{7}{2}} + 32119\delta^6 z_{\frac{7}{2},\frac{5}{2}})].$$

Third Panel-Second Panel.

$$\int_h^{2h} \int_{2k}^{3k} z\,dx\,dy = hk\,[z_{\frac{3}{2},\frac{5}{2}} + \tfrac{1}{24}(\delta^2 z_{\frac{3}{2},\frac{5}{2}} + \delta'^2 z_{\frac{3}{2},\frac{5}{2}}) - \tfrac{17}{5760}(\delta'^4 z_{\frac{3}{2},\frac{5}{2}} + 2\delta^4 z_{\frac{5}{2},\frac{5}{2}} - \delta^4 z_{\frac{7}{2},\frac{5}{2}})$$
$$+ \tfrac{1}{576}\delta^2\delta'^2 z_{\frac{3}{2},\frac{5}{2}} - \tfrac{17}{138240}(\delta^2\delta'^4 z_{\frac{3}{2},\frac{5}{2}} + \delta^4\delta'^2 z_{\frac{5}{2},\frac{5}{2}})$$
$$+ \tfrac{1}{967680}(367\delta'^6 z_{\frac{3}{2},\frac{7}{2}} - 2489\delta^6 z_{\frac{7}{2},\frac{5}{2}})].$$

Third Panel-Third Panel.

$$\int_{2h}^{3h} \int_{2k}^{3k} z\,dx\,dy = hk\,[z_{\frac{5}{2},\frac{5}{2}} + \tfrac{1}{24}(\delta^2 z_{\frac{5}{2},\frac{5}{2}} + \delta'^2 z_{\frac{5}{2},\frac{5}{2}}) - \tfrac{17}{5760}(\delta^4 z_{\frac{5}{2},\frac{5}{2}} + \delta'^4 z_{\frac{5}{2},\frac{5}{2}})$$
$$+ \tfrac{1}{576}\delta^2\delta'^2 z_{\frac{5}{2},\frac{5}{2}} - \tfrac{17}{138240}(\delta^2\delta'^4 z_{\frac{5}{2},\frac{5}{2}} + \delta^4\delta'^2 z_{\frac{5}{2},\frac{5}{2}})$$
$$+ \tfrac{367}{967680}(\delta'^6 z_{\frac{5}{2},\frac{7}{2}} + \delta^6 z_{\frac{7}{2},\frac{5}{2}})].$$

The remaining boundary formulae can be obtained from these by appropriate interchanges of suffixes, and there will be a similar arrangement at each corner of the rectangle ($x = 0$ to mh, $y = 0$ to nk) of integration.

These formulae might also be expressed in terms of ordinates only, but they seem to be more suitable, than the corner ordinate formulae, for working with differences, since only one ordinate and its differences, instead of four ordinates and their differences, are in general associated with each panel.

We will consider as an example the integral

$$\frac{1}{2\pi}\frac{1}{\sqrt{\cdot 96}}\int_0^{1\cdot 2}\int_0^{1\cdot 2} e^{-\frac{1}{1\cdot 92}(x^2 - \cdot 4xy + y^2)}\,dx\,dy.$$

This is the volume of the portion of the normal frequency surface

$$z = \frac{1}{2\pi}\frac{1}{\sqrt{1-r^2}}\,e^{-\frac{1}{2(1-r^2)}(x^2 - 2rxy + y^2)}$$

(for which r the correlation $= \cdot 2$) contained between the planes $x = 1\cdot 2$, $y = 1\cdot 2$ and the surface. We shall actually calculate

$$\int_0^{1\cdot 2}\int_0^{1\cdot 2} \frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{1\cdot 92}(x^2 - \cdot 4xy + y^2)}\,dx\,dy,$$

subsequently multiplying by $\dfrac{1}{\sqrt{2\pi(1-r^2)}}$, and shall use the ordinates at the 36 points given by $x = \cdot 1, \cdot 3, \cdot 5, \cdot 7, \cdot 9, 1\cdot 1$; $y = \cdot 1, \cdot 3, \cdot 5, \cdot 7, \cdot 9, 1\cdot 1$, working with ordinates to 5 figures. The following is a table of the 36 ordinates and the differences which are necessary:

TABLE VII. *Table of* $\dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{1\cdot 92}(x^2 - \cdot 4xy + y^2)}$ *and Central Differences.*

y		$x = \cdot 1$	$x = \cdot 3$	$x = \cdot 5$	$x = \cdot 7$	$x = \cdot 9$	$x = 1\cdot 1$
·1	z	·39563	·38107	·35207	·31199	·26520	·21622
	δ^2	—	− ·01444	− ·01108	− ·00671	− ·00219	—
	δ'^2	—	—	—	—	—	—
	δ^4	—	—	·00101	·00015	—	—
	δ'^4	—	—	—	—	—	—
	$\delta^2\delta'^2$	—	—	—	—	—	—
·3	z	·38107	·37011	·34481	·30812	·26410	·21713
	δ^2	—	− ·01434	− ·01139	− ·00733	− ·00295	—
	δ'^2	− ·01444	− ·01434	− ·01364	− ·01238	− ·01073	− ·00890
	δ^4	—	—	·00111	·00032	—	—
	δ'^4	—	—	—	—	—	—
	$\delta^2\delta'^2$	—	·00060	·00056	·00039	·00018	—
·5	z	·35207	·34481	·32391	·29187	·25227	·20914
	δ^2	—	− ·01364	− ·01114	− ·00756	− ·00353	—
	δ'^2	− ·01108	− ·01139	− ·01114	− ·01042	− ·00931	− ·00793
	δ^4	—	—	·00108	·00045	—	—
	δ'^4	·00101	·00111	·00108	·00106	·00102	·00089
	$\delta^2\delta'^2$	—	·00056	·00047	·00039	·00027	—
·7	z	·31199	·30812	·29187	·26520	·23113	·19322
	δ^2	—	− ·01238	− ·01042	− ·00740	− ·00384	—
	δ'^2	− ·00671	− ·00733	− ·00756	− ·00740	− ·00687	− ·00607
	δ^4	—	—	·00106	·00054	—	—
	δ'^4	·00015	·00032	·00045	·00054	·00055	·00052
	$\delta^2\delta'^2$	—	·00039	·00039	·00037	·00027	—
·9	z	·26520	·26410	·25227	·23113	·20312	·17123
	δ^2	—	− ·01073	− ·00931	− ·00687	− ·00388	—
	δ'^2	− ·00219	− ·00295	− ·00353	− ·00384	− ·00388	− ·00369
	δ^4	—	—	·00102	·00055	—	—
	δ'^4	—	—	—	—	—	—
	$\delta^2\delta'^2$	—	·00018	·00027	·00027	·00023	—
1·1	z	·21622	·21713	·20914	·19322	·17123	·14555
	δ^2	—	− ·00890	− ·00793	− ·00607	− ·00369	—
	δ'^2	—	—	—	—	—	—
	δ^4	—	—	·00089	·00052	—	—
	δ'^4	—	—	—	—	—	—
	$\delta^2\delta'^2$	—	—	—	—	—	—

Where blanks are left, the corresponding differences are unknown.

Using the appropriate mid-ordinate formula in the case of each panel, we have the following table of results:

TABLE VIII. *Table of Volumes over the Separate Panels.*

$$z = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{1\cdot 92}(x^2 - \cdot 4xy + y^2)}.$$

y	x: 0 – ·2	·2 – ·4	·4 – ·6	·6 – ·8	·8 – 1·0	1·0 – 1·2
0	·39563	·38107	·35207	·31199	·26520	·21622
	– ·00148	– ·00132	– ·00113	– ·00088	– ·00060	– ·00031
	·00014	·00007	·00006	·00006	·00006	·00002
·2	·39429	·37982	·35100	·31117	·26466	·21593
	·38107	·37011	·34481	·30812	·26410	·21713
	– ·00132	– ·00120	– ·00104	– ·00082	– ·00057	– ·00031
	·00007	– ·00001	– ·00001	– ·00001	·00000	– ·00002
·4	·37982	·36890	·34376	·30729	·26353	·21680
	·35207	·34481	·32391	·29187	·25227	·20914
	– ·00113	– ·00104	– ·00093	– ·00075	– ·00054	– ·00031
	·00006	– ·00001	– ·00001	·00000	·00000	– ·00001
·6	·35100	·34376	·32297	·29112	·25173	·20882
	·31199	·30812	·29187	·26520	·23113	·19322
	– ·00088	– ·00082	– ·00075	– ·00062	– ·00045	– ·00026
	·00006	– ·00001	·00000	·00000	·00000	·00000
·8	·31117	·30729	·29112	·26458	·23068	·19296
	·26520	·26410	·25227	·23113	·20312	·17123
	– ·00060	– ·00057	– ·00054	– ·00045	– ·00032	– ·00019
	·00006	·00000	·00000	·00000	·00000	·00000
1·0	·26466	·26353	·25173	·23068	·20280	·17104
	·21622	·21713	·20914	·19322	·17123	·14555
	– ·00031	– ·00031	– ·00031	– ·00026	– ·00019	– ·00011
	·00002	– ·00002	– ·00001	·00000	·00000	·00001
1·2	·21593	·21680	·20882	·19296	·17104	·14545

In the first line of each rectangular subdivision is given the central ordinate of the panel; the second line contains the terms in δ^2 and δ'^2 in the formula employed and the third line the terms in δ^4 and δ'^4. The fourth line is the sum of the first three and consequently gives the volume over the panel divided by ·04, the product of the respective distances between two consecutive ordinates in the x and y directions (the hk of our formulae). By summation of these results we can obtain the integral we require.

We find

Sum of volumes over all the panels ... = ·387984,

$$\frac{1}{\sqrt{2\pi}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots = \cdot 3989423,$$

$$\sqrt{\cdot 96} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots = \cdot 9797959,$$

whence
$$\frac{1}{2\pi}\frac{1}{\sqrt{\cdot 96}}\int_0^{1\cdot 2}\int_0^{1\cdot 2} e^{-\frac{1}{1\cdot 92}(x^2-\cdot 4xy+y^2)}\,dx\,dy = \cdot 15798.$$

By excluding from the summation the last two rows and the last column of the table we find

$$\frac{1}{\sqrt{2\pi}}\int_0^{1\cdot 0}\int_0^{\cdot 8} e^{-\frac{1}{1\cdot 92}(x^2-\cdot 4xy+y^2)}\,dx\,dy = \cdot 253186,$$

and
$$\frac{1}{2\pi}\frac{1}{\sqrt{\cdot 96}}\int_0^{1\cdot 0}\int_0^{\cdot 8} e^{-\frac{1}{1\cdot 92}(x^2-\cdot 4xy+y^2)}\,dx\,dy = \cdot 10309,$$

which is in agreement with the value of the same integral found by a different method on p. 51.

We may check the former of these two results by the use of tetrachoric functions as follows:

Let Q denote the value of the integral, then

$$Q = \{\tfrac{1}{2} - \tau_0(1\cdot 2)\}^2 + \{\tau_1(0) - \tau_1(1\cdot 2)\}^2 r + \ldots + \{\tau_n(0) - \tau_n(1\cdot 2)\}^2 r^n + \ldots,$$

where r is in this case ·2, and

$$\tau_n(x) = \frac{1}{\sqrt{\underline{|n}}}\left(-\frac{d}{dx}\right)^{n-1}\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}.$$

We have the following numerical values*:

n	$\tau_n(0)$	$\tau_n(1\cdot 2)$	$\tau_n(0)-\tau_n(1\cdot 2)$	$\{\tau_n(0)-\tau_n(1\cdot 2)\}^2$	r^n	$\{\tau_n(0)-\tau_n(1\cdot 2)\}^2 r^n$
0	·5	·11507	·38493	·14817	1	·14817
1	·39894	·19418	·20476	·04193	·2	·00839
2	0	·16477	− ·16477	·02715	·04	·00109
3	− ·16287	·03335	− ·19622	·03850	·008	·00031
4	0	− ·07420	·07420	·00551	·0016	·00001
5	·10925	− ·06322	·17247	·02975	·00032	·00001
6	0	·02403	− ·02403	·00058	·000064	·00000
						Q = ·15798

This is in agreement with the above result for Q.

* From the tables of tetrachoric functions. *Tables for Statisticians*, Table XXIX.

The cubature method with mid-ordinates seems well adapted for calculating the *theoretical* frequency within each cell of a correlation table, when a frequency surface has been fitted to the bivariate distribution, in order to apply a 'goodness-of-fit' test. It will of course be necessary to calculate the ordinates of the theoretical surface at the central points of the cells; but with the exception of the rather unlikely case of the function being exactly integrable we are bound to resort to some method of cubature and on the whole it is more convenient to use one central ordinate and its differences rather than four corner ordinates and their differences.

E. Integration over an Area with a Curved Boundary.

We will conclude this paper by the consideration of a numerical example in which the preceding methods are not applicable, owing to one or both of the limits of integration for one variable being a function of the other variable, or to the integration being taken over an area with a curved boundary. In such a case we have to fall back upon the method indicated in the introduction and perform a series of quadratures, in each of which one variable is kept constant and, finally, to perform another quadrature on the results.

We shall consider

$$U=\int_6^{7\cdot2}\int_0^{3[1-(x/7\cdot2)^2]}\left\{1-\left(\frac{x}{7\cdot2}\right)^2-\frac{y}{3}\right\}dx\,dy.$$

This is the volume of the portion of the parabolic cylinder $z=1-\left(\frac{x}{7\cdot2}\right)^2-\frac{y}{3}$, contained by the planes $z=0, x=6$, $y=0$ and the surface and can be calculated by direct integration.

We have purposely chosen a volume that admits of direct integration, in order to be able to check the accuracy of our result.

Let x be given the values

$x_1=6\cdot0,\quad x_2=6\cdot2,\quad x_3=6\cdot4,\quad x_4=6\cdot6,\quad x_5=6\cdot8,\quad x_6=7\cdot0,\quad x_7=7\cdot2,$

and let $\gamma_1, \gamma_2, \gamma_3 \ldots \gamma_7$ be the corresponding values of

$$\int_0^{3[1-(x/7\cdot2)^2]}\left\{1-\left(\frac{x}{7\cdot2}\right)^2-\frac{y}{3}\right\}dy.$$

We propose first to calculate $\gamma_1, \gamma_2, \gamma_3 \ldots \gamma_7$ by Weddle's formula, using seven equidistant ordinates for each, and then to calculate $U=\int_6^{7\cdot2}\gamma dx$ by using Weddle's formula again on $\gamma_1, \gamma_2, \gamma_3 \ldots \gamma_7$.

This involves dividing each of the seven values of $3\left[1-\left(\frac{x}{7\cdot2}\right)^2\right]$ corresponding to $x=6\cdot0$, $6\cdot2 \ldots 7\cdot2$ into six equal parts, and calculating the ordinates of the surface at the points of division.

We first obtain the corresponding values of x and $3\left[1-\left(\frac{x}{7\cdot2}\right)^2\right]$, viz.:

x	$3[1-(x/7\cdot2)^2]$	x	$3[1-(x/7\cdot2)^2]$
6·0	·9166668	6·8	·3240741
6·2	·7754631	7·0	·1643520
6·4	·6296295	7·2	0
6·6	·4791666		

Next we obtain the values of y, in the plane $z=0$, at which we must calculate the ordinates of the surface:

$x=6\cdot0$	$x=6\cdot2$	$x=6\cdot4$	$x=6\cdot6$	$x=6\cdot8$	$x=7\cdot0$	$x=7\cdot2$
0	0	0	0	0	0	0
·1527778	·1292439	·1049383	·0798611	·0540124	·0273920	0
·3055556	·2584877	·2098765	·1597222	·1080247	·0547840	0
·4583334	·3877316	·3148148	·2395833	·1620371	·0821760	0
·6111112	·5169754	·4197530	·3194444	·2160494	·1095680	0
·7638890	·6462193	·5246913	·3993055	·2700618	·1369600	0
·9166668	·7754631	·6296295	·4791666	·3240741	·1643520	0

Then from the relation $z=1-\frac{x^2}{51\cdot84}-\frac{y}{3}$ we find the ordinates at these points:

$x=6\cdot0$	$x=6\cdot2$	$x=6\cdot4$	$x=6\cdot6$	$x=6\cdot8$	$x=7\cdot0$	$x=7\cdot2$
·3055556	·2584877	·2098765	·1597222	·1080247	·0547840	0
·2546297	·2154064	·1748971	·1331017	·0900206	·0456533	0
·2037037	·1723251	·1399177	·1064814	·0720165	·0365227	0
·1527778	·1292438	·1049382	·0798610	·0540123	·0273920	0
·1018519	·0861626	·0699588	·0532406	·0360082	·0182613	0
·0509259	·0430813	·0349794	·0266203	·0180041	·0091307	0
0	0	0	0	0	0	0

Weddle's formula is

$$\int_0^{6h} u_x dx = \frac{3h}{10}\{u_0 + 5u_h + u_{2h} + 6u_{3h} + u_{4h} + 5u_{5h} + u_{6h}\}.$$

Applying it to each of the above sets of equidistant ordinates we have

$\gamma_1 = \cdot1400463$, $\gamma_2 = \cdot1002239$, $\gamma_3 = \cdot0660722$, $\gamma_4 = \cdot0382667$,
$\gamma_5 = \cdot0175040$, $\gamma_6 = \cdot0045019$, $\gamma_7 = 0$,

and applying Weddle's formula again to the γ's we reach $U = \cdot0586111$.

Now it may be shown by direct integration that

$$\begin{aligned} U &= \frac{3}{2}\left\{\frac{8}{15}(7{\cdot}2) - 6 + \frac{2}{3}\frac{6^3}{(7{\cdot}2)^2} - \frac{1}{5}\frac{6^5}{(7{\cdot}2)^4}\right\} \\ &= \frac{3}{2}\left\{3{\cdot}84 - 6 + \frac{4}{1{\cdot}44} - \frac{6}{5}\frac{1}{(1{\cdot}44)^2}\right\} \\ &= {\cdot}0586111, \end{aligned}$$

so that our quadrature is correct to seven figures.

The computer, however, will be in error if he imagines that he will always obtain such an accurate result with the same ease; the differences of the ordinates are of course the criterion. In this case, in each of the first set of quadratures, the first differences are constant and, in the final quadrature, fourth differences are constant so that Weddle's formula, which is correct up to and including fifth order differences, gives an accurate result.

To emphasise the point let us now consider

$$V = \int_6^{7\cdot2}\int_0^{3\sqrt{1-(x/7\cdot2)^2}} \left\{1 - \left(\frac{x}{7{\cdot}2}\right)^2 - \left(\frac{y}{3}\right)^2\right\} dx dy,$$

which is the volume of the portion of the paraboloid $z = 1 - \left(\frac{x}{7{\cdot}2}\right)^2 - \left(\frac{y}{3}\right)^2$ between the planes $z = 0$, $x = 6$, $y = 0$ and the surface. Treating it in the same way as the previous integral, we first obtain the corresponding values of x and $3\sqrt{1-\left(\frac{x}{7{\cdot}2}\right)^2}$:

x	$3\sqrt{1-(x/7{\cdot}2)^2}$	x	$3\sqrt{1-(x/7{\cdot}2)^2}$
6·0	1·6583124	6·8	·9860133
6·2	1·5252506	7·0	·7021795
6·4	1·3743684	7·2	0
6·6	1·1989578		

Next we obtain the values of y, in the plane $z = 0$, at which we must calculate the ordinates:

$x=6{\cdot}0$	$x=6{\cdot}2$	$x=6{\cdot}4$	$x=6{\cdot}6$	$x=6{\cdot}8$	$x=7{\cdot}0$	$x=7{\cdot}2$
0	0	0	0	0	0	0
·2763854	·2542084	·2290614	·1998263	·1643356	·1170299	0
·5527708	·5084169	·4581228	·3996526	·3286711	·2340598	0
·8291562	·7626253	·6871842	·5994789	·4930067	·3510897	0
1·1055416	1·0168337	·9162456	·7993052	·6573422	·4681196	0
1·3819270	1·2710422	1·1453070	·9991315	·8216778	·5851496	0
1·6583124	1·5252506	1·3743684	1·1989528	·9860133	·7021795	0

Then from the relation $z = 1 - \frac{x^2}{51{\cdot}84} - \frac{y^2}{9}$ we find the ordinates at these points:

$x=6{\cdot}0$	$x=6{\cdot}2$	$x=6{\cdot}4$	$x=6{\cdot}6$	$x=6{\cdot}8$	$x=7{\cdot}0$	$x=7{\cdot}2$
·3055556	·2584877	·2098765	·1597222	·1080247	·0547840	0
·2970679	·2513075	·2040466	·1552855	·1050240	·0532622	0
·2716050	·2297668	·1865569	·1419753	·0960220	·0486969	0
·2291667	·1938658	·1574074	·1197916	·0810185	·0410880	0
·1697531	·1436043	·1165981	·0887346	·0600137	·0304356	0
·0933642	·0789823	·0641289	·0488040	·0330075	·0167395	0
0	0	0	0	0	0	0

Applying Weddle's formula to each of the above sets of equidistant ordinates, we have

$$\gamma_1 = {\cdot}3378044, \quad \gamma_2 = {\cdot}2628390, \quad \gamma_3 = {\cdot}1922984, \quad \gamma_4 = {\cdot}1276605,$$
$$\gamma_5 = {\cdot}0710092, \quad \gamma_6 = {\cdot}0256455, \quad \gamma_7 = 0,$$

and applying Weddle's formula again to the γ's, we reach $V = {\cdot}1685699$.

Now it may be shown by direct integration that

$$V = \tfrac{1}{8}(7{\cdot}2)(3)\pi - 1{\cdot}8\left(\tfrac{1}{4}\sin 4\theta + 2\sin 2\theta + 3\theta\right),$$

where $\sin\theta = \frac{1}{1{\cdot}2} = {\cdot}8\dot{3}$.

$$\therefore\ \theta = 56^\circ\, 26'{\cdot}5614 \text{ and } \tfrac{1}{4}\sin 4\theta = -\ {\cdot}1791387,$$
$$2\sin 2\theta = 1{\cdot}8425693,$$
$$3\theta = 2{\cdot}9553323,$$
$$\tfrac{1}{8}(7{\cdot}2)(3)\pi = 8{\cdot}4823002.$$
$$\therefore\ V = 8{\cdot}4823002 - 1{\cdot}8\,(4{\cdot}6187629)$$
$$= {\cdot}1685270.$$

Thus our result by quadrature is in error by more than 4 in the fifth decimal place, although we took the ordinates to seven decimal places. This is due to the fact that γ is now a function whose differential coefficients, except the first, become infinite at $x = 7{\cdot}2$* and consequently its differences diverge there. This difficulty is generally met with at a point, after passing which the function becomes imaginary as is the case with γ in this example for $x > 7{\cdot}2$.

The difficulty may be, to some extent, surmounted by increasing the number of ordinates. We have calculated the values of γ for the additional values

* It can be shown by integration that $\gamma = 2\left\{1 - \left(\frac{x}{7{\cdot}2}\right)^2\right\}^{\frac{3}{2}}$.

$x = 6{\cdot}1,\ 6{\cdot}3,\ 6{\cdot}5\ \ldots\ 7{\cdot}1$ so that we can now use Weddle's formula with 13 ordinates,

$$\int_0^{12h} u_x dx = \frac{3h}{10}\{u_0 + u_{2h} + u_{4h} + u_{8h} + u_{10h} + u_{12h} + 2u_{6h} + 5(u_h + u_{5h} + u_{7h} + u_{11h}) + 6(u_{3h} + u_{9h})\}.$$

Denoting our new set of γ's by $\gamma_1, \gamma_2 \ldots \gamma_{13}$ we have

$\gamma_1 = {\cdot}3378044$	$\gamma_6 = {\cdot}1591331$	$\gamma_{10} = {\cdot}0466168$
$\gamma_2 = {\cdot}2998465$	$\gamma_7 = {\cdot}1276605$	$\gamma_{11} = {\cdot}0256455$
$\gamma_3 = {\cdot}2628390$	$\gamma_8 = {\cdot}0981770$	$\gamma_{12} = {\cdot}0091630$
$\gamma_4 = {\cdot}2269325$	$\gamma_9 = {\cdot}0710092$	$\gamma_{13} = 0$
$\gamma_5 = {\cdot}1922984$		

and applying Weddle's formula for 13 ordinates we have

$$V = {\cdot}1685343.$$

The result is now correct to five places, which is an improvement, but to get it correct to six places we should almost certainly require 25 γ's which would make the work very laborious.

In fact, no general rule can be given for getting over a difficulty of this kind. Each case must be considered on its own merits, and sometimes a special artifice may be found.

For instance, in the present case we might proceed as follows:

The integral is $\iint \left[1 - \left(\frac{x}{7{\cdot}2}\right)^2 - \left(\frac{y}{3}\right)^2\right] dxdy$ over the portion of the ellipse $\left(\frac{x}{7{\cdot}2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$ bounded by the lines $x = 6$ and $y = 0$ and the curve.

Put $\quad x = 7{\cdot}2X$ and $y = 3Y$.

This is equivalent to changing the ellipse into a circle by an appropriate 'stretch and squeeze.' The integral becomes

$$21{\cdot}6 \iint [1 - X^2 - Y^2]\, dX\, dY$$

over the portion of the circle $X^2 + Y^2 = 1$ bounded by the lines

$$Y = 0,\ X = \frac{6}{7{\cdot}2} = {\cdot}8\dot{3}.$$

Now changing into polar coordinates, this becomes

$$21{\cdot}6 \iint (1 - r^2)\, r\, d\theta\, dr$$

over the same area, which is shown shaded in the diagram.

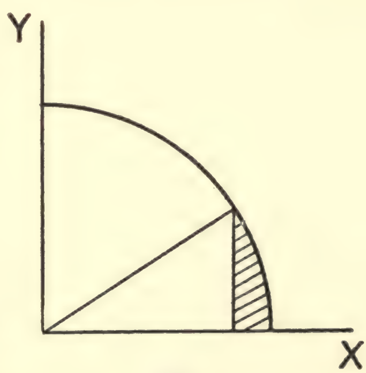


Fig. 4

This can be written

$$21{\cdot}6\left\{\int_{r=0}^{1}\int_{\theta=0}^{\alpha}(1-r^2)\,r\,d\theta\,dr-\int_{\theta=0}^{\alpha}\int_{r=0}^{\cdot 8\dot{3}\sec\theta}(1-r^2)\,r\,dr\,d\theta\right\},$$

where $\alpha=\cos^{-1}\cdot 8\dot{3}=33^\circ{\cdot}55730954$

$$=21{\cdot}6\alpha\left\{\int_{r=0}^{1}(r-r^3)\,dr\right\}-21{\cdot}6\int_{\theta=0}^{\alpha}d\theta\int_{r=0}^{\cdot 8\dot{3}\sec\theta}(r-r^3)\,dr$$

$$=21{\cdot}6\,\frac{\alpha}{4}-21{\cdot}6\int_{\theta=0}^{\alpha}\left\{\frac{(\cdot 8\dot{3})^2\sec^2\theta}{2}-\frac{(\cdot 8\dot{3})^4\sec^4\theta}{4}\right\}d\theta,$$

where α is, of course, in circular measure, or

$$\alpha=\cdot 58568554^{*}.$$

We shall now calculate this integral by the use of Weddle's formula with 13 ordinates. Dividing the range between 0° and 33°·55730954 into 12 equal intervals, we have the following table of angles and ordinates:

θ in degrees	$\sec\theta$	$\frac{(\cdot 8\dot{3})^2}{2}\sec^2\theta-\frac{(\cdot 8\dot{3})^4}{4}\sec^4\theta$
0	1·000000000	·226658950
2·79644246	1·001192251	·226911279
5·59288492	1·004783259	·227665271
8·38932739	1·010816211	·228909438
11·18576985	1·019364495	·230623357
13·98221231	1·030533911	·232773449
16·77865477	1·044465934	·235307621
19·57509723	1·061342223	·238146786
22·37153969	1·081390642	·241172204
25·16798216	1·104893210	·244206517
27·96442462	1·132196484	·246985264
30·76086708	1·163725167	·249113620
33·55730954	1·200000000	·250000000

h, the interval between the ordinates, in circular measure, is given by $h=\cdot 048807128$, whence, using Weddle's formula,

$$\int_{\theta=0}^{\alpha}\left\{\frac{(\cdot 8\dot{3})^2\sec^2\theta}{2}-\frac{(\cdot 8\dot{3})^4\sec^4\theta}{4}\right\}d\theta=\cdot 138619199,$$

and the value we are seeking

$$=3{\cdot}1627019-2{\cdot}9941747$$
$$=\ \cdot 1685272.$$

* Here we have actually integrated for r. We could, of course, have used a quadrature formula to calculate the integral with regard to r for chosen values of θ. Weddle's formula would have given an exact result, since the function of r is only of the third degree.

This result differs from the true answer by 2 in the seventh place, and is correct to 6 places.

After reading this paper, the computer may perhaps come to the conclusion that cubature is a laborious and somewhat tentative process.

It is true that only experience, a study of the differences and consideration of the accuracy required in particular cases, can guide him in deciding what number of ordinates he should take. But at any rate when the limits of integration are constants his procedure is fairly well defined. If he wishes to work to a very great degree of accuracy, he will find one of the general formulae of Table V to suit him, or he may, using differences, work with the formulae of Section D. If considerations of time and labour make brevity of importance, he can, by combining the results we have given for the different panels, find for himself an appropriate formula. But if his limits of integration are not constants and cannot be made so by any simple transformation—and this may of course apply to cases where the mathematical form of the function is unknown—a study of our last example may suggest to him some way of surmounting his difficulties.

I am indebted to Miss Ida McLearn for the preparation of the diagrams.

APPENDIX: BIBLIOGRAPHY

I. QUADRATURE

(1) I. NEWTON: *Methodus Differentialis*, pp. 93—101 of the *Analysis per Quantitatum, Series, Fluxiones ac Differentias*....London, 1711. Gives, in addition to the interpolation formulae generally attributed to Stirling and Bessel, a method of finding high-order parabolae through any number of given points, shows how to use them for purposes of quadrature and suggests and starts the construction of tables of the necessary coefficients, which were actually completed by Cotes.

(2) D. COTES: *Harmonia Mensurarum*. Cambridge, 1722. At the end of this work there is a tract entitled *De methodo differentiali Newtoniana* in which Newton's method is given for finding high-order parabolae through given points and applying them to find quadrature formulae depending on equidistant ordinates. A table of results is given for each case from three ordinates up to eleven ordinates, based respectively on parabolae of the 2nd, 3rd ... 10th orders.

(3) L. EULER: *Methodus generalis Summandi Progressiones*, *Commentarii Academiae scientiarum imperialis Petropolitanae*, 1739, read in the year 1732 but not published until 1739 as the Petersburg Academy was behind with its publications. Gives the well-known Euler-McLaurin formula. In view of the fundamental nature of this theorem in the theory of quadrature, particularly in determining the error of any given quadrature formula, we venture to give Euler's and also McLaurin's proofs of the formula. Euler's is as follows:

If $s(n)$ be the sum of a series of n terms, then by Taylor's theorem

$$s(n) - s(n-1) = \frac{ds}{dn} - \frac{1}{\underline{|2}}\frac{d^2s}{dn^2} + \frac{1}{\underline{|3}}\frac{d^3s}{dn^3} - \dots,$$

or if t be the nth term,

$$t = \left(\frac{d}{dn} - \frac{1}{\underline{|2}}\frac{d^2}{dn^2} + \frac{1}{\underline{|3}}\frac{d^3}{dn^3} - \dots\right)s,$$

or by inverting the series of differential operators,

$$s = \frac{1}{\dfrac{d}{dn}\left\{1 - \dfrac{1}{\underline{|2}}\dfrac{d}{dn} + \dfrac{1}{\underline{|3}}\dfrac{d^2}{dn^2} - \dots\right\}}\,t,$$

or

$$s = \int t\,dn + \frac{t}{2} + \frac{1}{12}\frac{dt}{dn} - \frac{1}{720}\frac{d^3t}{dn^3} + \frac{1}{30240}\frac{d^5t}{dn^5} - \dots,$$

which does not make quite clear the question of the limits between which the differential coefficients have to be taken. Euler, however, applies the formula to some examples quite correctly.

(4) C. McLaurin: *A Treatise on Fluxions.* Edinburgh, 1742. Gives, Vol. II, p. 672, the Euler-McLaurin formula both in its tangential and chordal forms, i.e. (with a change from McLaurin's notation)

$$\int_0^{ph} u_x dx = h\left(\tfrac{1}{2}u_0 + u_h + u_{2h} + \ldots + \tfrac{1}{2}u_{ph}\right) - \left[\frac{1}{12}h^2\frac{du}{dx} - \frac{h^4}{720}\frac{d^3u}{dx^3} + \frac{h^6}{30240}\frac{d^5u}{dx^5} - \ldots\right]_0^{ph},$$

and

$$\int_0^{ph} u_x dx = h\left(u_{\frac{1}{2}h} + u_{\frac{3}{2}h} + u_{\frac{5}{2}h} + \ldots + u_{(p-\frac{1}{2})h}\right) + \left[\frac{1}{24}h^2\frac{du}{dx} - \frac{7h^4}{5760}\frac{d^3u}{dx^3} + \frac{31h^6}{967680}\frac{d^5u}{dx^5} - \ldots\right]_0^{ph}.$$

The latter form, as far as we know, was not given by Euler. McLaurin's proof of the first form is in substance as follows:

Let $A = \int_a^{a+h} f(x)\,dx$, then

$$A = hf(a) + \frac{h^2}{\underline{|2}}f'(a) + \frac{h^3}{\underline{|3}}f''(a) + \ldots,$$

$$f(a+h) - f(a) = hf'(a) + \frac{h^2}{\underline{|2}}f''(a) + \frac{h^3}{\underline{|3}}f'''(a) + \ldots,$$

$$f'(a+h) - f'(a) = hf''(a) + \frac{h^2}{\underline{|2}}f'''(a) + \frac{h^3}{\underline{|3}}f^{\text{iv}}(a) + \ldots,$$

etc. etc.

Then he eliminates the coefficients on the right-hand side except the first, obtaining:

$$A - \frac{h}{2}\{f(a+h) - f(a)\} + \frac{h^2}{12}\{f'(a+h) - f'(a)\} - \frac{h^4}{720}\{f'''(a+h) - f'''(a)\}$$
$$+ \frac{h^6}{30240}\{f^{\text{v}}(a+h) - f^{\text{v}}(a)\} - \text{etc.} = hf(a),$$

or $$A = \frac{h}{2}\{f(a+h) + f(a)\} - \frac{h^2}{12}\{f'(a+h) - f'(a)\} + \frac{h^4}{720}\{f'''(a+h) - f'''(a)\} - \text{etc.}$$

This establishes the theorem for two ordinates, and by summation McLaurin extends it to any number.

Neither in the *Phil. Trans.* between the years 1713 and 1743 nor elsewhere, have we found any reference to McLaurin's having published this theorem before 1742. It therefore seems that Euler must be awarded the priority, though from the fact that Euler obtained it from the point of view of the sum of a series, and McLaurin from the opposite point of view of finding an area, we can be almost certain that each, unaware of the other's work, obtained the theorem independently.

(5) T. Simpson: *Mathematical Dissertations*, p. 109. London, 1743. Gives a geometrical proof of the formula usually known by his name, i.e.

$$\int_0^{2nh} f(x)\,dx = \frac{h}{3}\left[u_0 + u_{2nh} + 2\{u_{2h} + u_{4h} + \ldots + u_{2(n-1)h}\} + 4\{u_h + u_{3h} + \ldots + u_{(2n-1)h}\}\right],$$

in which it is necessary to have an odd number of ordinates.

(6) C. F. Gauss: "Methodus nova integralium valores per approximationem inveniendi," first published in 1816, *Werke*, III, pp. 165—196. Göttingen, 1866. Gauss first gives Cotes' formulae of which he says, "quas praeunte summo Newton evolutas dedit Cotes," and discusses the error of the approximations.

He then proves his celebrated theorem that n properly chosen ordinates suffice to give the integral $\int_a^b f(x)\,dx$ exactly, provided $f(x)$ is a polynomial of degree not higher than the $2n-1$th. He finds how the ordinates have to be chosen and he gives tables of distances apart of ordinates and of coefficients for the use of the method. We have seen (Part I of this paper) that Gauss' method generally gives, for mathematical functions, better results for the same number of ordinates than equal-interval formulae, and these results are particularly good when the function can be expanded in a convergent series of ascending powers of x, but in many practical cases the method will be valueless owing to the impossibility of obtaining the ordinates at the points required by the theory.

The Gaussian method has given rise to a number of investigations, which are primarily of theoretical interest.

(7) C. G. J. Jacobi: "Über Gauss' Neue Methode die Werthe der Integrale Näherungsweise zu finden," *Werke*, VI, p. 8, 1891. Jacobi gives an investigation, first published in 1826, of the Gaussian method, and the error when $f(x)$ is of degree higher than $2n-1$, and shows that the polynomial $\phi(x)$, whose roots give the points at which the ordinates have to be taken, is given by

$$\phi(x) = C\frac{d^n}{dx^n}(x-a)^n(x-b)^n;$$

actually he uses the limits 0 and 1, but the argument is the same. $\phi(x)$ reduces for the limits -1 and 1 to a constant multiplied into one of Legendre's polynomials.

(8) J. W. Lubbock: "On the comparison of various tables of annuities," *Cambridge Phil. Trans.* III, p. 322, 1829. Gives, with applications to annuities, the formula usually known as Lubbock's for the sum of a large number of terms of a series, viz.

$$\{y_0 + y_i + y_{2i} + \dots + y_{m-i}\} = n\{y_0 + y_1 + \dots + y_{m-1}\}$$
$$+\frac{n-1}{2}\{y_m - y_0\} - \frac{n^2-1}{12n}\{\Delta y_m - \Delta y_0\} + \frac{n^2-1}{24n}\{\Delta^2 y_m - \Delta^2 y_0\} - \dots,$$

where $ni = 1$. This, of course, reduces to the difference form of the Euler-McLaurin theorem when i is indefinitely diminished.

(9) T. Parmentier: "Comparaisons de quelques méthodes de quadrature et formule nouvelle," *Nouvelles annales de mathématiques*, 1855. Discusses the

ordinary tangential and chordal quadrature formulae, Simpson's formula, and Poncelet's formula, viz.

$$\int_0^{nh} y\,dx = h\left[y_{\frac{1}{2}} + y_{\frac{3}{2}} + \dots + y_{n-\frac{1}{2}}\right] - \tfrac{1}{8}h\left\{y_{\frac{1}{2}} - y_0 + y_{n-\frac{1}{2}} - y_n\right\}.$$

He points out that the uncorrected tangential formula is, in general, better than the chordal and Poncelet's better than either. Then he gives the formula generally known under his own name, viz.

$$\int_0^{nh} y\,dx = h\left[y_{\frac{1}{2}} + y_{\frac{3}{2}} + \dots + y_{n-\frac{1}{2}}\right] - \tfrac{1}{12}h\left\{y_{\frac{1}{2}} - y_0 + y_{n-\frac{1}{2}} - y_n\right\},$$

which he shows is, in general, better than Poncelet's, and thinks is as accurate as Simpson's. It is not however a very good formula, though it may sometimes be useful for rough data where the differences are irregular.

(10) E. B. Christoffel: "Über die Gaussische Quadratur und eine Verallgemeinerung derselben," *Crelle's Journal*, LV, p. 61 (1858), shows that it is possible by the use of $m + n$ ordinates, of which n are given and the remainder properly chosen, to find the integral $\int_{-1}^{1} f(x)\,dx$ exactly, provided $f(x)$ is a polynomial of degree not higher than $2m + n - 1$. He shows that $\phi(x) = C\dfrac{d^m}{dx^m}\left[(x^2-1)^m V\right]$ where V is a polynomial determined by the n given abscissae.

(11) E. Heine: *Handbuch der Kugelfunctionen*, Berlin, 1861, contains a good discussion of the Gaussian method of quadrature and the error involved.

(12) E. Heine: "Mittheilung über Kettenbrüche," *Crelle's Journal*, LXVII, p. 315, 1867. This work, taken in conjunction with the previous one, will show the reader the connection between the generalised Gaussian method for approximating to $\int_a^b f(x)\,\psi(x)\,dx$, $\psi(x)$ being a known function, and the theory of expanding functions in continued fractions. In it Heine shows that if the integral $\displaystyle\int_a^b \frac{\psi(x)\,dx}{x-z}$ be expanded as a continued fraction, the denominator of the nth convergent $\phi_n(z)$ is a semi-orthogonal function, i.e. $\int_a^b \phi_n(x)\,\phi_m(x)\,\psi(x)\,dx = 0$, when $m \neq n$ and $\int_a^b \phi_n(x)\,\psi(x)\,x^h\,dx = 0$ for $h = 0, 1 \dots n-1$, and that these conditions determine ϕ uniquely provided ψ does not change its sign in the interval a to b; and further that the numerator of the convergent is $\displaystyle\int_a^b \frac{\phi_n(x) - \phi(z)}{x-z}\,\psi(x)\,dx$.

Now if we wish to approximate to $\int_a^b f(x)\,\psi(x)\,dx$ by means of substituting for $f(x)$ the Lagrange formula

$$F(x) = \sum_{i=1}^{i=n} \frac{f(a_i)}{\phi'(a_i)}\,\frac{\phi(x)}{(x-a_i)},$$

where $\phi(x) = (x - a_1)(x - a_2) \ldots (x - a_n)$, what are the best values of the a's to take? Suppose $f(x)$ to be a polynomial of degree $2n - 1$ and let

$$f(x) = Q(x)\,\phi(x) + R(x).$$

Then $Q(x)$ is of degree $n - 1$ at most, also $R(x)$, and

$$\int_a^b f(x)\,\psi(x)\,dx = \int_a^b Q(x)\,\phi(x)\,\psi(x)\,dx + \int_a^b R(x)\,\psi(x)\,dx.$$

Accordingly if $\phi(x)$ is chosen so that $\int_a^b x^h\,\phi(x)\,\psi(x)\,dx = 0$ for $h = 0, 1 \ldots (n-1)$ we have

$$\int_a^b f(x)\,\psi(x)\,dx = \int_a^b R(x)\,\psi(x)\,dx,$$

and $$f(x) = R(x) \text{ for } x = a_1, a_2 \ldots a_n.$$

Therefore by choosing our ordinates in this way, we get as good an approximation with n ordinates as we should otherwise with $2n$.

Further, if we write our approximate value as $\Sigma A_1 f(a_i)$,

$$A_i = \int_a^b \frac{1}{\phi'(a_i)} \frac{\phi(x)\,\psi(x)\,dx}{x - a_i}$$

$$= \int_a^b \frac{1}{\phi'(a_i)} \frac{\{\phi(x) - \phi(z)\}\,\psi(x)\,dx}{x - z}\ (z = a_i).$$

Hence using Heine's result, if $\phi_n(z)$ and $Z_n(z)$ are respectively the denominator and numerator of the nth convergent of $\int_a^b \frac{\psi(x)\,dx}{x - z}$ expanded as a continued fraction, the approximate value of the integral $\int_a^b f(x)\,\psi(x)\,dx$ will be

$$\sum_{i=1}^{i=n} \frac{Z_n(a_i)}{\phi'_n(a_i)} f(a_i).$$

Two examples of this may perhaps be given here.

(α) If we take $\psi(x) = 1$, $a = -1$, $b = 1$, we have

$$\phi_n(x) = CP_n(x),$$

$$A_i = \int_a^b \frac{P_n(x)}{P'_n(a_i)} \frac{dx}{(x - a_i)},$$

which is Gauss' original case.

(β) If we take $\psi(x) = \frac{1}{\sqrt{1 - x^2}}$, $a = -1$, $b = 1$, we have

$$\phi_n(x) = \cos n\,(\cos^{-1} x),$$

or if we write $x = \cos\theta$, $a_i = \cos\lambda_i$,

$$A_i = \frac{1}{n\dfrac{\sin n\lambda_i}{\sin\lambda_i}} \int_0^\pi \frac{\cos n\theta - \cos n\lambda_i}{\cos\theta - \cos\lambda_i}\,d\theta,$$

but
$$\int_0^\pi \frac{\cos n\theta - \cos n\lambda_i}{\cos\theta - \cos\lambda_i}\,d\theta = \frac{\pi \sin n\lambda_i}{\sin\lambda_i},$$

$$\therefore\ A_i = \frac{\pi}{n}.$$

Whence $\int_0^\pi f(\cos\theta)\,d\theta = \frac{\pi}{n}\left\{f\left(\cos\frac{\pi}{2n}\right) + f\left(\cos\frac{3\pi}{2n}\right) + \ldots + f\cos\frac{(2n-1)\pi}{2n}\right\}$,

or
$$\int_{-1}^1 \frac{f(x)\,dx}{\sqrt{1-x^2}} = \frac{\pi}{n}\{f(a_1) + f(a_2) + \ldots + f(a_n)\},$$

where $a_1, a_2 \ldots a_n$ are the roots of $\cos n(\cos^{-1} x) = 0$. This approximate formula is given by several writers (see below) and may occasionally be useful.

(13) F. MEHLER: "Bemerkungen zur Theorie der mechanischen Quadraturen," *Crelle's Journal*, LXIII, p. 152, 1863, considers integrals of the form

$$\int_{-1}^1 (1-x)^\lambda (1+x)^\mu f(x)\,dx,$$

and shows that the position of the ordinates should be given by

$$(1-x)^{-\lambda}(1+x)^{-\mu}\frac{d^n}{dx^n}\{(1-x)^{n+\lambda}(1+x)^{n+\mu}\} = 0$$

in order to get the greatest possible accuracy. The case $\lambda = \mu = -\frac{1}{2}$ leads to the above formula for $\int_0^\pi f(\cos\theta)\,d\theta$. As far as we know Mehler was the first to obtain it.

(14) W. S. B. WOOLHOUSE: *On Interpolation, Summation and the Adjustment of Numerical Tables.* London, 1865. See also *The Assurance Magazine* and *Journal of the Institute of Actuaries*, Vol. XI, pp. 61—88, 301—332; Vol. XII, pp. 136—176. London, 1864—66. This is largely occupied with the applications of an individual formula to quadrature.

(15) G. BOOLE (Edn. MOULTON): *A Treatise on the Calculus of Finite Differences.* London, 1872. Gives, in Chap. III, a discussion of quadrature and references which are still valuable.

(16) C. HERMITE: *Cours d'Analyse de l'École Polytechnique*, Première Partie (Paris, 1873), pp. 439—455, "Évaluation approchée des intégrales." Contains a discussion of the Gaussian method and gives the above formula for $\int_0^\pi f(\cos\theta)\,d\theta$, with references to the papers of Mehler and Christoffel quoted above. Tchebycheff, in a paper on quadrature (1874, see below), refers to this formula as if Hermite had discovered it, although Mehler had given it ten years earlier and Hermite had seen his work.

(17) M. Chévilliet: *Comptes rendus des séances de l'académie des sciences*, T. LXXVIII, p. 1841. Paris (1874). "Sur le degré d'exactitude de la formule de Simpson, relative à l'évaluation des aires." Gives the error in using Simpson's formula as $-\frac{h^4}{180}\{f'''(X)-f'''(x_0)\}$ + terms of higher order, where the ordinates are taken at intervals of h between the points $x=x_0$ and $x=X$, thus incidentally showing the formula to be exact for a cubic.

(18) P. Tchebycheff: "Sur les quadratures," *Liouville's Journal*, 2nd Series, Vol. XIX, p. 19, 1874. Finds the position of the ordinates in order that

$$\int_{-1}^{1} F(x)\,\phi(x)\,dx$$

may be approximately represented by

$$k\,[\phi(x_1)+\phi(x_2)+\ldots+\phi(x_n)].$$

He finds that $k=\frac{1}{n}\int_{-1}^{1}F(x)\,dx$ and that $x_1, x_2 \ldots x_n$ are the roots of the equation $f(z)=0$, where $f(z)$ is the polynomial part of

$$e^{\frac{1}{k}\int_{-1}^{1}F(x)\log(z-x)\,dx}.$$

By taking $F(x)=1/\sqrt{1-x^2}$ he gets Mehler's formula for $\int_0^\pi f(\cos\theta)\,d\theta$. He gives tables for the case $F(x)=1$ when $k=2/n$. His method has the advantage over Gauss' that all his coefficients are the same, but the error being of the order $\phi^{n+1}(0)$, we require twice as many ordinates for the same accuracy as Gauss'. Tchebycheff points out that if the ordinates are themselves subject to errors, his method has the advantage that all the errors are equally weighted.

We have tried Tchebycheff's method on $\int_0^1 \frac{1}{1+x}\,dx$ with seven ordinates. We find, working to six figures only (he only gives his abscissae to six figures), $\log_e 2 = \cdot 693147$, which is correct to the last figure; but when we tried the method on $\int_0^3 \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\,dx$ we obtained ·4986982. The correct answer is ·4986501, Simpson's rule gives ·4986237, Weddle's ·4986614 and Gauss' ·4986498, all with seven ordinates; so that Tchebycheff's method is the worst. Thus we cannot recommend it for mathematical functions, and if we are dealing with observed quantities, it would be almost certainly impossible to obtain them at the intervals required by the formula.

(19) E. Gourier: *Comptes rendus des séances de l'académie des sciences*, T. XCVII, p. 79. Paris, 1884. "Sur une méthode capable de fournir une valeur approchée de l'intégrale $\int_{-\infty}^{\infty} F(x)\,dx$." Shows that to approximate most exactly to the integral

$$\int_{-\infty}^{\infty} f(x)\,e^{-\frac{1}{2}x^2}\,dx,$$

if n ordinates are used, their abscissae should be chosen to satisfy

$$e^{x^2/2} \frac{d^{n+1}}{dx^{n+1}} (e^{-x^2/2}) = 0.$$

(20) R. Radau : *Comptes rendus des séances de l'académie des sciences*, T. XCVII, p. 157. Paris, 1884. "Remarque sur le calcul d'une intégrale définie." Shows that for $\int_{-\infty}^{\infty} f(x) e^{-x} dx$, the appropriate position of the ordinates is given by

$$e^x \frac{d^n}{dx^n} (x^n e^{-x}) = 0.$$

(21) A. Markoff : *Mathematische Annalen*, p. 428, 1884, "Sur la méthode de Gauss pour le calcul approché des intégrales." Markoff arrives at the generalised Gaussian formula for $\int_a^b f(x) \psi(x) dx$, by using a polynomial $F(x)$ of degree $2n-1$ which touches $f(x)$ at n points ($x = a_1, a_2 \ldots a_n$). Whence he deduces the approximate formula

$$\int_a^b f(x) \psi(x) dx = \Sigma A_i f(a_i) + \Sigma B_i f'(a_i),$$

and gives the forms of A, B and $F(x)$. Then he chooses the a's in such a manner that all the B's vanish. He thus arrives at the generalised Gaussian formula given above, and shows that the error can be expressed in the form

$$\frac{f^{2n}(\xi)}{\lfloor n} \int_a^b [\phi(x)]^2 \psi(x) dx,$$

where $$\phi(x) = (x - a_1)(x - a_2) \ldots (x - a_n)$$

and $a < \xi < b$, provided $\psi(x)$ does not change its sign between $x = a$ and $x = b$.

He deduces Mehler's formula given above and uses it to calculate

$$\eta = \frac{1}{\pi} \int_{-1}^{1} \frac{\log(11 + y) dy}{\sqrt{1 - y^2}}$$

and $$\zeta = \frac{1}{\pi} \int_{-1}^{1} \frac{dy}{\sqrt{1 - y^2} \log(11 + y)},$$

and gets the former integral right to eight and the latter to six significant figures.

(22) H. L. Rice : *The Theory and Practice of Interpolation.* Lynn, Mass. 1899. Gives a number of quadrature formulae based on Newton's, Stirling's and Bessel's interpolation formulae; the notation, however, seems to us unnecessarily complicated.

(23) W. F. Sheppard: "Some Quadrature Formulae," *Proceedings of the London Mathematical Society*, Vol. XXXII, 1900. Gives a number of very useful quadrature formulae, together with a discussion of the error they involve.

The quadrature formulae he gives are of two kinds; the first kind, of the same type as Weddle's and Simpson's, is obtained by considering such expressions as

$$A_f = fh\left(\tfrac{1}{2} z_0 + z_f + z_{2f} + \ldots + z_{m-f} + \tfrac{1}{2} z_m\right)$$

(where f is a factor of m) for different values of f, and by summing such multiples of these expressions as make one or more of the differential coefficients in the Euler-McLaurin formula vanish, and so obtaining formulae of great accuracy. In the second type the corrective terms are so chosen that the first term in the Euler-McLaurin expansion which is neglected is as small as possible. See Part I of this paper, pp. 5—7.

(24) GEORGE KING: *Text Book of the Institute of Actuaries*, Part II, Second Edition. London, 1902. Gives a number of quadrature formulae particularly useful to actuaries, with illustrations from actuarial data.

(25) G. F. BECKER: *Phil. Mag.* 1911, "Some New Mechanical Quadratures." In this paper the writer, by summing appropriate multiples of

$$T = 2h\{y_1 + y_3 + y_5 + \dots + y_{n-1}\}$$

and

$$C_m = mh\left\{\frac{y_0}{2} + y_m + y_{2m} + \dots + \frac{y_n}{2}\right\}$$

for various values of m, eliminates one or more of the early differential coefficients in the Euler-McLaurin expansion, and so obtains quadrature formulae of great accuracy. But since $2C_1 - C_2 = T$, these formulae are identical with those found by Sheppard in his 1900 paper, and all of them, with one exception, are, in fact, given in that paper.

(26) DAVID GIBB: "A course in Interpolation and Numerical Integration," *Edinburgh Mathematical Tracts*, No. 2. London, 1915. This book has a very good chapter on quadrature, putting in a form, easily comprehensible, most of the well-known quadrature formulae; including those of Woolhouse and Lubbock for expressing the sum of a large number of ordinates in terms of a few ordinates with a wider interval between them.

II. CUBATURE

There is not very much literature on cubature. We give, however, the following references:

(1) F. MINDING: "Über die Berechnung des Näherungswerthes doppelter Integrale," *Crelle's Journal*, VI. Berlin, 1829. Gives an extension to two dimensions of the Gaussian Quadrature method, of which the substance is as follows:

Let $z = \phi(x, y)$ be a polynomial of the $2n-1$th degree in x and the $2m-1$th in y; let $P_n(x)$, $P_m(y)$ be Legendre's polynomials of the nth and mth orders respectively in x and y; then dividing z by $P_n(x)$, $P_m(y)$ we obtain

$$z = z' + \lambda P_n(x) + \mu P_m(y) + \nu P_n(x) P_m(y),$$

where z' is of the $n-1$th, $m-1$th degrees (at most) in x and y and λ, ν are of degree $n-1$ at most in x, μ of degree $m-1$ at most in y. Therefore:

$$\int_{-1}^{1}\int_{-1}^{1}\lambda P_n(x)\,dxdy=0,\quad \int_{-1}^{1}\int_{-1}^{1}\mu P_m(y)\,dxdy=0,\quad \int_{-1}^{1}\int_{-1}^{1}\nu P_n(x)\,P_m(y)\,dxdy=0.$$

$$\therefore\quad \int_{-1}^{1}\int_{-1}^{1} z\,dxdy=\int_{-1}^{1}\int_{-1}^{1} z'\,dxdy.$$

Moreover if the roots of $P_n(x)$ are $a_1, a_2 \ldots a_n$ and of $P_m(y)$ are $\alpha_1, \alpha_2 \ldots \alpha_n$, $z=z'$ for the mn values $x=a_1, a_2 \ldots a_n$, $y=\alpha_1, \alpha_2 \ldots \alpha_n$.

Again $$z'=P_n(x)\,P_m(y)\sum_{r=1}^{n}\sum_{s=1}^{m}\frac{1}{P_n'(a_r)\,P_m'(\alpha_s)}\,\frac{\phi(a_r, \alpha_s)}{(x-a_r)\,(y-\alpha_s)}.$$

Hence we may find a quadrature formula giving

$$\int_{-1}^{1}\int_{-1}^{1} z'\,dxdy \text{ and therefore } \int_{-1}^{1}\int_{-1}^{1} z\,dxdy$$

correctly, and depending only on the above mn ordinates.

(2) P. Appell: "Sur une classe de polynômes à deux variables et le calcul approché des intégrales doubles." *Annales de la faculté des Sciences de Toulouse*, T. IV, 1890. In this paper Appell discusses the approximate calculation of double integrals of the form $\iint k(x,y)f(x,y)\,dxdy$, $k(x,y)$ being a known function. The method employed is an extension to two dimensions of the generalised Gaussian method; he assumes $f(x, y)$ can be expanded in powers of x and y and that k is of the same sign throughout the region of integration. The discussion is entirely theoretical. He shows that $\iint f(x, y)\,dxdy$ taken over the circle $x^2+y^2=1$ is approximately given by $\frac{\pi}{3}\{f(x_1, y_1)+f(x_2, y_2)+f(x_3, y_3)\}$, where x_1, y_1, x_2, y_2, x_3, y_3 are three points on the circle $x^2+y^2=\frac{1}{2}$ forming an equilateral triangle; the approximation being correct up to and including 2nd order terms in $f(x, y)$.

(3) J. Clerk Maxwell: "On Approximate Multiple Integration between limits by Summation." *Proceedings of the Cambridge Philosophical Society*, Vol. III, p. 40, 1880. (Read March 12th, 1877.) The Gaussian method of quadrature is extended to double integrals of the type $I=\int_{x_1}^{x_2}\int_{y_1}^{y_2} u\,dxdy$, where u is of the form $f(a+bx+cy)$, which by the transformation

$$x=\tfrac{1}{2}(x_1+x_2)+\tfrac{1}{2}p\,(x_2-x_1),$$
$$y=\tfrac{1}{2}(y_1+y_2)+\tfrac{1}{2}q\,(y_2-y_1)$$

becomes $$\tfrac{1}{4}(x_2-x_1)(y_2-y_1)\int_{-1}^{1}\int_{-1}^{1} u\,dpdq.$$

If f is a polynomial of degree not greater than $2n-1$, Maxwell's method gives the

integral exactly by the use of only n values of u, n being odd. The method however would be rather cumbersome to apply for more than three or five ordinates. Maxwell's formula for three values of u is as follows:

$$I = \tfrac{1}{4}(x_2 - x_1)(y_2 - y_1)\{R_0 u_0 + R_1(u_1 + u_1')\},$$

where
$$R_0 = \frac{4}{3}\frac{\beta^4 + 5\beta^2\gamma^2 + \gamma^4}{3\beta^4 + 10\beta^2\gamma^2 + 3\gamma^4}, \quad R_1 = \frac{5}{6}\frac{\beta^4 + 2\beta^2\gamma^2 + \gamma^4}{3\beta^4 + 10\beta^2\gamma^2 + 3\gamma^4},$$

$$\beta = \tfrac{1}{2}b(x_2 - x_1), \quad \gamma = \tfrac{1}{2}c(y_2 - y_1),$$

$$u_0 = f(r_0) = f\left[a + \frac{b}{2}(x_2 + x_1) + \frac{c}{2}(y_2 + y_1)\right],$$

$$u_1 = f(r_0 + \zeta_1), \quad u_1' = f(r_0 - \zeta_1),$$

where
$$\zeta_0 = 0, \quad \zeta_1 = \pm\frac{(3\beta^4 + 10\beta^2\gamma^2 + \gamma^4)^{\frac{1}{2}}}{(5\beta^2 + 5\gamma^2)^{\frac{1}{2}}}.$$

Maxwell then considers the case when the form of u is unrestricted and finds the following formula for I, exact when u is any function of x and y of degree not greater than seven:

$$I = \tfrac{1}{4}(x_2 - x_1)(y_2 - y_1)\{Pu_{00} + Q\Sigma(u_{p,0}) + R\Sigma u_{q,r}\},$$

where
$$\Sigma(u_{p,0}) = u_{p,0} + u_{0,p} + u_{-p,0} + u_{0,-p},$$

$$\Sigma(u_{q,r}) = u_{q,r} + u_{q,-r} + u_{-q,r} + u_{-q,-r}$$
$$+ u_{r,q} + u_{r,-q} + u_{-r,q} + u_{-r,-q},$$

$$P = \tfrac{8}{162}, \quad Q = \tfrac{98}{162}, \quad R = \tfrac{31}{162},$$

$$p^2 = \tfrac{12}{35}, \quad q^2 = \tfrac{3}{5}\left[1 + \left(\tfrac{6}{31}\right)^{\frac{1}{2}}\right], \quad r^2 = \tfrac{3}{5}\left[1 - \left(\tfrac{6}{31}\right)^{\frac{1}{2}}\right],$$

or
$$p = \cdot5855400, \quad q = \cdot9294971, \quad r = \cdot5796854$$

(these numerical values are given wrongly in the paper).

We tried this method on the integral

$$W = \frac{1}{2\pi}\int_0^1\int_0^{\cdot8}\frac{1}{\sqrt{\cdot96}}e^{-\frac{1}{1\cdot92}(x^2 - \cdot4xy + y^2)}\,dx\,dy$$

(cf. p. 58), which becomes, by the transformation

$$x = \cdot5(1 + p), \quad y = \cdot4(1 + q),$$

$$\frac{(\cdot2)}{\sqrt{2\pi}}\frac{1}{\sqrt{\cdot96}}\int_{-1}^1\int_{-1}^1\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left\{\sqrt{\frac{1}{\cdot96}\{\cdot25p^2 + \cdot16q^2 - \cdot08pq + \cdot42p + \cdot24q + \cdot33\}}\right\}^2}\,dp\,dq,$$

whence the ordinates can be calculated by means of tables of the probability integral and Maxwell's method applied. We find $W = \cdot10309$ which is in agreement to five places with the result on p. 58. Thus it appears that this method, which only requires the use of thirteen ordinates, may well be useful for mathematical functions.

Maxwell concludes his paper with a quadrature formula for triple integrals on the same principle, where the region of integration is a parallelepiped.

The quadrature is based on the values of the function at 27 points in space, and is exact if the function is of degree not greater than seven.

Two sets of abscissae and coefficients arise, but, in each set, some of the points are outside the region of integration, so that the method is only available for mathematical functions and not observational data. This is the only quadrature formula for triple integrals that we have seen.

(4) H. L. RICE: *The Theory and Practice of Interpolation.* Lynn, Mass. 1899. Considers the particular case of $\iint_{T'}^{T''} F(T)\,dT^2$, where the indefinite integral has an assigned value at the lower limit of integration.

(5) W. F. SHEPPARD: "Some Quadrature Formulae," *Proceedings of the London Mathematical Society*, Vol. XXXII, 1900. This paper also gives a short discussion of cubature formulae for the integral $V=\int_0^{mh}\int_0^{nk} z\,dx\,dy$, that is to say a volume on a rectangular base which is supposed to be divided by $m-1$ planes at distance h in one direction and $n-1$ at distance k in the other. Dr Sheppard points out that the given ordinates may either be

(i) The edges of the constituent prisms,

(ii) The mid-ordinates of the faces in one direction,

(iii) The central ordinates of the prisms.

He regards the first as being the most convenient and discusses it in some detail. He uses the same method as in his first type of quadrature formulae. Let V_{11} denote

$$hk\left\{\begin{array}{lllll} \frac{1}{4}z_{00} & +\frac{1}{2}z_{10} & +\frac{1}{2}z_{20} & +\ldots & +\frac{1}{4}z_{m,0} \\ +\frac{1}{2}z_{01} & +\ z_{11} & +\ z_{21} & +\ldots & +\frac{1}{2}z_{m,1} \\ +\frac{1}{2}z_{02} & +\ z_{12} & +\ z_{22} & +\ldots & +\frac{1}{2}z_{m,2} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ +\frac{1}{2}z_{0,n-1} & +\ z_{1,n-1} & +\ z_{2,n-1} & +\ldots & +\frac{1}{2}z_{m,n-1} \\ +\frac{1}{4}z_{0,n} & +\frac{1}{2}z_{1,n} & +\frac{1}{2}z_{2,n} & +\ldots & +\frac{1}{4}z_{m,n} \end{array}\right\},$$

and $V_{r,s}$ denote the same expression when the ordinates are at intervals rh, sk instead of h, k; r, s being factors respectively of m and n.

Then by summing appropriate multiples of such expressions as $V_{r,s}$, Dr Sheppard shows it is possible to get rid of any number of the boundary differential coefficients in the two-dimensional form of the Euler-McLaurin formula, and so to obtain quadrature formulae just as in the univariate case.

He gives two examples:

$$(1)\quad V=\tfrac{1}{9}(16V_{11}-4V_{21}-4V_{12}+V_{22}),$$

which is the two-dimensional form of Simpson's formula, and

$$(2)\quad V=\tfrac{1}{450}\{960V_{11}-384V_{12}+64V_{13}-300V_{21}+120V_{22} \\ -20V_{23}+15V_{41}-6V_{42}+V_{43}\}.$$

We tried the former of these on the integral W (cf. Example, p. 51) and again found $W=\cdot 10309$.

NOTE. Correction of Certain Formulae in Tract III.

The reader will find that formulae (α), (β) facing p. 49 in the above Tract require the following modifications, which he is requested to make in his copy of that Tract, should he possess one. The correct values are used on pp. 44, 46 of Tract X.

Line 4,	1st curled bracket	*for* $(1+\theta)\,\delta^2\delta'^2 z_{12}$ *read* $(1+\theta)\,\delta^2\delta'^2 z_{11}$.
,,	2nd curled bracket	*for* $-(1+\theta)\,\delta^2\delta'^2 z_{12}$ *read* $(1+\theta)\,\delta^2\delta'^2 z_{12}$.
Line 6,		*for* $+\frac{1}{240}$ *read* $-\frac{1}{240}$.
Line 11,	1st term	*for* $-\frac{1}{720}$ *read* $-\frac{1}{144}$.
Last line,		*for* 'eighth' *read* 'seventh.'

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